# An algebraic Birkhoff decomposition for the continuous renormalization group

P. Martinetti

Università di Roma, *La Sapienza*

MPIM Bonn, July 2007

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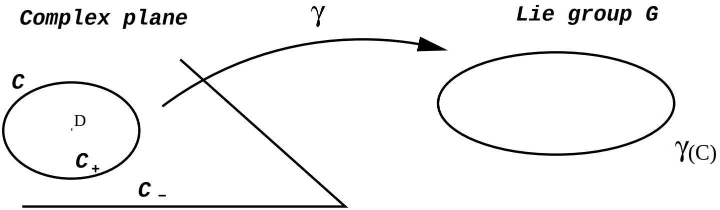
- Perturbative renormalization in qft is a Birkhoff decomposition
  $\longrightarrow$ Hopf algebra of Feynman diagrams.(*Connes-Kreimer* 2000)
- Exact renormalization is an *algebraic* Birkhoff decomposition
  $\longrightarrow$ Hopf algebra of decorated rooted trees.

# Program

- Birkhoff decomposition
- Exact Renormalization Group equations as fixed point equation
- Power series of trees
- Algebraic Birkhoff decomposition for the ERG

Collaboration with T. Krajewski and F. Girelli
*An algebraic Birkhoff decomposition for the continuous renormalization group*, J. Math. Phys. **45** (2004) 4679-4697, hep-th/0401157.

# Birkhoff decomposition



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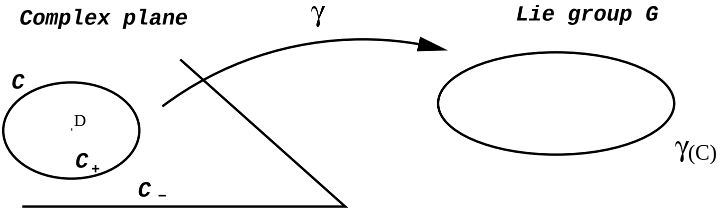


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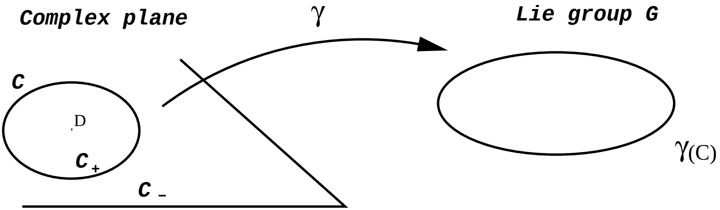


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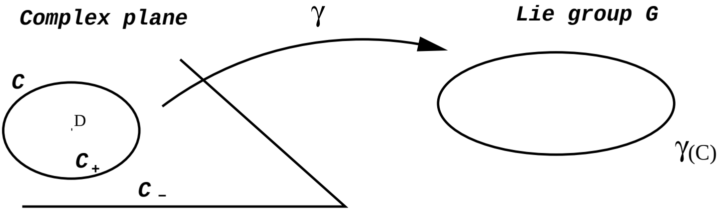


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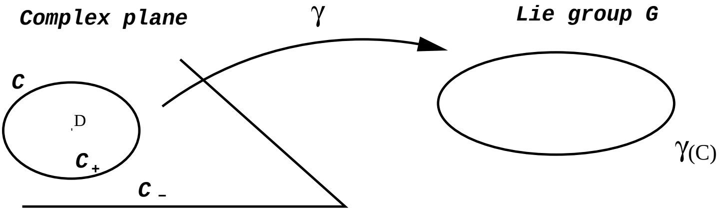


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$\to$ dimensional regularization in QFT: $D$ is the dimension of space time, $G$ is the group of characters of the Hopf algebra of Feynman diagrams.

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$\mathcal{A}$ : complex functions in $\mathbb{C}$, pole in $D$ (=4).
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$\mathcal{A}_-$: polynômial in $\frac{1}{z-D}$ without constant term.

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Compose with character $\chi_z$ of $\mathcal{A}$,

$$\gamma(z) \doteq \chi_z \circ U, \quad \gamma_-(z) \doteq \chi_z \circ C, \quad \gamma_+(z) \doteq \chi_z \circ R,$$

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The renormalized theory is the evaluation at $D$ of the positive part of the Birkhoff decomposition of the bare theory.

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<u>Definition</u>(*Connes, Kreimer, Kastler*): $H$ commutative Hopf algebra, $\mathcal{A}$ commutative algebra. $p_-$ projection onto a subalgebra $\mathcal{A}_-$.
An algebra morphism $\gamma : H \to \mathcal{A}$ has a unique <u>algebraic Birkhoff decomposition</u> if there exist two algebra morphisms $\gamma_+$, $\gamma_-$ from $H$ to $\mathcal{A}$ such that

$$\gamma_+ = \gamma_- * \gamma$$

$$p_+ \gamma_+ = \gamma_+, \quad p_- \gamma_- = \gamma_-$$

with $p_+$ the projection on

$$\mathcal{A}_+ = \text{Ker } p_-.$$

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Dimensional analysis : $\Lambda \to t$, $S \to x$, $\beta \mapsto X$,

$$\frac{\partial x}{\partial t} = Dx + X(x)$$

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- $x(t)$ represents the parameters at a scale $t$.
- $\tilde{x}_0$ encodes the initial conditions at a fixed scale $t_0$.

# ERG as fixed point equation: mixed initial conditions

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Renormalization deals with change of initial condition in fixed point equation.

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- Coordinate free notations: $\chi'(\chi)$ is the map $\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$

$$y \mapsto \chi'_y(\chi(y)).$$

# Power series of trees: characters of the Hopf algebra

$\chi^{\emptyset} \doteq \mathbb{I}, \quad \chi^{\bullet} \doteq \chi, \quad \chi$  $\doteq \chi'(\chi), \quad \chi$  $\doteq \frac{1}{2}\chi''(\chi,\chi)\ldots$

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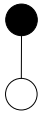
Taylor expansion:

$\chi(\mathbb{I}+\chi) = \chi^{\bullet} + \chi$ $+ \chi$ $+ \ldots$

$$= \sum_T \phi(T)\chi^T$$

$$= f_\phi[\chi]$$

where $\phi(T) = 1$ for any rooted tree $T$, except $\phi(\emptyset) = 0$.

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| |
|---|
| Formal power series starting with $\mathbb{I}$, i.e. $\phi(\emptyset) = 1 \iff$ opposite group of characters of the Hopf algebra of rooted trees. |

*Butcher group, B-series*

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- $x = x_R + \chi_R(x) \Longrightarrow x = f_{\phi_1}[\chi_R](x_R)$
- $\xi \doteq \mathbb{I} - (\mathbb{I} - \chi_R) \circ (\mathbb{I} - \chi_0)^{-1} \Longrightarrow (\mathbb{I} - \chi_R)^{-1} = (\mathbb{I} - \chi_0)^{-1} \circ (\mathbb{I} - \xi)^{-1}$

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| | | |
|---|---|---|
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# Power series of trees: solution of fixed point equation

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$$Y^\bullet = \xi, \qquad Y^{\blacksquare} = \chi_R$$

$\Phi_-(H_\bullet) = 1, \quad \Phi_+(H_\blacksquare) = 1, \quad \Phi_- = \Phi_+$ otherwise.

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$$\boxed{\gamma_+ = \gamma_- * \gamma}$$

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