Kantorovich metric in noncommutative geometry

Pierre Martinetti
Università di Napoli *Federico II*

QP 33
CIRM, Luminy 5th October 2012

Introduction: noncommutative geometry in a nutshell

Any commutative C^*-algebra is isomorphic to an algebra of continuous functions vanishing at infinity on some topological space $\mathcal{P}(\mathcal{A})$, Gelfand duality

$$\mathcal{A} \simeq C_0(\mathcal{P}(\mathcal{A})), \quad \mathcal{P}(C_0(\mathcal{X})) \simeq \mathcal{X}.$$

$\mathcal{P}(\mathcal{A})$ is the set of pure states of $\mathcal{A}$, i.e. the extremal points of the set $\mathcal{S}(\mathcal{A})$ of normalized ($\mathbb{I} \to 1$), positive ($a^* a \to \mathbb{R}^+$) linear maps $\mathcal{A} \to \mathbb{C}$:

$$\mathcal{P}(C_0(\mathcal{X})) \ni \delta_x : x \to f(x) \qquad S(C_0(\mathcal{X})) \ni \varphi : f \to \int_{\mathcal{X}} f \, \mathrm{d}\mu$$

Connes' theory of spectral triples $(\mathcal{A}, \mathcal{H}, D)$ extends Gelfand duality beyond topology, so that to encompass differential, homological, metric (spin) aspects,

$$\begin{array}{rcl} \text{commutative spectral triple} & \longrightarrow & \text{noncommutative spectral triple} \\ \updownarrow & & \downarrow \\ \text{Riemannian geometry} & & \text{non-commutative geometry} \end{array}$$

- Geometry without points, but the latter are retrieved as pure states of $\mathcal{A}$.
- How does one retrieve the Riemannian distance on $\mathcal{P}(C_0(\mathcal{M})) \simeq \mathcal{M}$ a Riemannian manifold, and extend it to $\mathcal{P}(\mathcal{A})$ for noncommutative $\mathcal{A}$?

Outline:

I. The metric aspect of noncommutative geometry

- spectral distance in noncommutative geometry
- Monge-Kantorovich distance in optimal transport theory: $\mathcal{A} = C_0(\mathcal{M})$

II. Towards a theory of optimal transport in noncommutative geometry?

- spectral distance on pure states as a cost function
- product of the continuum by the discrete: $\mathcal{A} = C_0(\mathcal{M}) \otimes \mathbb{C}^2$

III. Metric aspect of the Moyal plane: $\mathcal{A} = \mathbb{K}$

I. The metric aspect of noncommutative geometry

Spectral triple

An involutive algebra $\mathcal{A}$, a faithful representation on $\mathcal{H}$, an operator D on $\mathcal{H}$ such that $[D, a]$ is bounded and $a[D - \lambda\mathbb{I}]^{-1}$ is compact for any $a \in \mathcal{A}$ and $\lambda \notin$ Sp D.

When a set of conditions (dimension, regularity, finitude, first order, orientability) is satisfied, then

Theorem Connes 1996-2008

$\mathcal{M}$ a compact Riemann manifold, then $(C^\infty(\mathcal{M}), \Omega^\bullet(\mathcal{M}), d + d^\dagger)$ is a spectral triple.

When $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact Riemannian manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

Whatever $\mathcal{A}$, commutative or not, one defines on its state space $\mathcal{S}(\mathcal{A})$ the spectral distance (possibly infinite)

$$d_D(\varphi, \tilde{\varphi}) = \sup_{a \in \mathcal{A}} \{ |\varphi(a) - \tilde{\varphi}(a)| \,/\, \|[D, a]\| \leq 1 \}.$$

Optimal transport

$\mathcal{X}$ a locally compact Polish space, $c(x, y)$ a positive real function, the "cost". The minimal work W required to transport the probability measure μ_1 to μ_2 is

$$W(\mu_1, \mu_2) \doteq \inf_{\pi} \int_{\mathcal{X} \times \mathcal{X}} c(x, y) \, \mathrm{d}\pi$$

where the infimum is over all transportation plans, i.e. measures π on $\mathcal{X} \times \mathcal{X}$ with marginals μ_1, μ_2.

When the cost function c is a distance d, then

$$W(\mu_1, \mu_2) \doteq \inf_{\pi} \int_{\mathcal{X} \times \mathcal{X}} d(x, y) \, \mathrm{d}\pi$$

is a distance (possibly infinite) on the space of probability measures on $\mathcal{X}$, called the Monge-Kantorovich or Wasserstein distance of order 1.

Proposition

Rieffel 99, then D'Andrea, P.M. 2009

Let $\mathcal{X} = \mathcal{M}$ be a complete, connected, without boundary, Riemannian manifold. For any $\varphi, \tilde{\varphi} \in \mathcal{S}(C_0(\mathcal{M}))$,

$$W(\varphi, \tilde{\varphi}) = d_D(\varphi, \tilde{\varphi})$$

where W is the Monge-Kantorovich distance associated to the cost d_{geo}, while d_D is the spectral distance associated to $\left(C_0^\infty(\mathcal{M}), \Omega^\bullet(\mathcal{M}), D = d + d^\dagger\right)$.

i. Kantorovich duality:

$$W(\varphi, \tilde{\varphi}) = \sup_{\|f\|_{\mathrm{Lip}} \leq 1} \left(\int_{\mathcal{X}} f \, \mathrm{d}\mu - \int_{\mathcal{X}} f \, \mathrm{d}\tilde{\mu} \right)$$

with supremum on all real 1-Lipschitz $f \in C(\mathcal{X})$: $|f(x) - f(y)| \leq d_{\text{geo}}(x, y)$.

ii. For $f = f^*$, $\left\|[d + d^\dagger, f]^2\right\| = ||[\partial\!\!\!/, f]||^2 = \frac{1}{2} \|[[\Delta, f], f]\| = \|f\|^2_{\mathrm{Lip}}$.

iii. Any 1-Lip. f non-vanishing at infinity can be approximated by the 1-Lip.

$$f_n(x) \doteq f(x) e^{-d_{\text{geo}}(x_0, x)/n} \in C_0(\mathcal{M});$$

and any f_n is the uniform limit of a sequence of smooth 1-Lip. functions.

II. Towards a theory of optimal transport in noncommutative geometry ?

d_D commutative case	$\longrightarrow$	d_D noncommutative case
$\uparrow$		$\vert$
Kantorovich duality		Kantorovich duality ?
$\downarrow$		$\downarrow$
W with $d_D(\delta_x, \delta_y)$ as a cost function		W_D for some noncommutative cost ?
	$\searrow$	
		Wasserstein metric for free probability, *Biane*, *Voiculescu*.

Let $\mathcal{A}$ be a separable C^*-algebra with unit and $\varphi \in \mathcal{S}(\mathcal{A})$. There exists a (non-necessarily unique) probability measure $\mu \in \text{Prob}(\mathcal{P}(\mathcal{A}))$ such that

$$\varphi(a) = \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\mu(\omega) \quad \text{where} \quad \hat{a}(\omega) \doteq \omega(a).$$

Optimal transport on $\mathcal{P}(\mathcal{A})$, with cost d_D, yields a distance W_D on $\mathcal{S}(\mathcal{A})$, whose dual formulation is

$$W_D(\varphi, \tilde{\varphi}) \doteq \sup_{a \in \text{Lip}_D(\mathcal{A})} \left\{ \left| \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\mu(\omega) - \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\tilde{\mu}(\omega) \right| \right\}.$$

where

$$\text{Lip}_D(\mathcal{A}) \doteq \{a \in \mathcal{A} \text{ such that } |\omega_1(a) - \omega_2(a)| \leq d_D(\omega_1, \omega_2)\ \forall \omega_1, \omega_2 \in \mathcal{P}(\mathcal{A})\}.$$

Proposition P.M. 2011

For any $\varphi, \tilde{\varphi} \in \mathcal{S}(\mathcal{A})$, $d_D(\varphi, \tilde{\varphi}) \leq W_D(\varphi, \tilde{\varphi})$.

- Obvious because $\{a \in \mathcal{A}, \|D, a\| \leq 1\} \subset \text{Lip}_D(\mathcal{A})$.
- If $d_D = W_D$, noncommutative geometry coud be seen as a “provider” of cost functions.

A two-point space:

$$\mathcal{A} = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}$$

where $m \in \mathbb{C}$ and representation

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

This is a two-point space

$$\delta_1(z_1, z_2) \doteq z_1, \quad \delta_2(z_1, z_2) \doteq z_2$$

with distance

$$d_D(\delta_1, \delta_2) = \frac{1}{|m|}.$$

- Discrete space (i.e. no geodesic) but finite distance.
- For non pure states, $d_D = W_D$ since

$$\mathrm{Lip}_D(\mathbb{C}^2) = \left\{ a \in \mathbb{C}^2, \, |z_1 - z_2| \leq \frac{1}{|m|} \right\} = \left\{ a \in \mathbb{C}^2, \, \|[D, a]\| \leq 1 \right\}.$$

Product of the continuum by the discrete

Product of a manifold $\mathcal{M}$ by $\left(\mathbb{C}^2, \mathbb{C}^2, D_I = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}\right)$, namely

$$\mathcal{A}' = C_0^\infty(\mathcal{M}) \otimes \mathbb{C}^2,\ \mathcal{H}' = \Omega^\bullet(\mathcal{M}) \otimes \mathbb{C}^2,\ D' = (d + d^\dagger) \otimes \mathbb{I}_2 + \Gamma \otimes D_I.$$

Proposition

P.M., Wulkenhaar 2001

The spectral distance $d_{D'}$ between pure states of $\overline{\mathcal{A}'} = C_0(\mathcal{M}) \otimes \mathbb{C}^2$,

$$\mathcal{P}(\overline{\mathcal{A}'}) \simeq \mathcal{M} \cup \mathcal{M} = \left\{x_i \doteq (x, \delta_i),\ x \in \mathcal{M}, \delta_i \in \mathcal{P}(\mathbb{C}^2)\right\},$$

coincides with the geodesic distance in $\mathcal{M}' = \mathcal{M} \times [0,1]$ with Riemannian metric

$$\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \frac{1}{|m|} \end{pmatrix}.$$

- Possible to make m a function on $\mathcal{M}$: Higgs field in the standard model of elementary particles.

$\mathcal{S}(\overline{\mathcal{A}'})$ is the set of couples of measures (μ, ν) on $\mathcal{M}$, normalized to

$$\int_{\mathcal{M}} d\mu + \int_{\mathcal{M}} d\nu = 1,$$

whose evaluation on $\mathcal{A}' \ni a = (f, g)$, with $f, g \in C_0^\infty(\mathcal{M})$, is

$$\varphi(a) = \int_{\mathcal{M}} f\, d\mu + \int_{\mathcal{M}} g\, d\nu.$$

- As before, $d_{D'}(\varphi, \tilde{\varphi}) \leq W_{D'}(\varphi, \tilde{\varphi})$ where $W_{D'}$ is the Kantorovich distance on $\mathcal{M} \cup \mathcal{M}$ associated to the cost $d_{D'}$.

- Equality holds - $d_{D'} = d_D = W_D$ - for states localized on the same copy:

$$\varphi = (0, \nu),\ \tilde{\varphi} = (0, \tilde{\nu}) \quad \text{or} \quad \varphi = (\mu, 0),\ \tilde{\varphi} = (\tilde{\mu}, 0).$$

- For two states localized on distinct copies, one may project back the problem on a single copy, using a cost function defined solely on $\mathcal{M}$,

$$c(x, y) \doteq d_{D'}(x_1, y_2) \doteq \sqrt{d(x, y)^2 + \frac{1}{|m|^2}}.$$

  The Higgs field would then represent the cost to stay at the same point of space-time, but jumping from one copy to the other: $c(x, x) = \frac{1}{|m|} \neq 0$.

III. Noncommutative geometry of the Moyal plane

$$\mathcal{A} = (\mathcal{S}(\mathbb{R}^2), \star), \quad \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, \quad D = -i \sum_{\mu=1}^{2} \sigma^\mu \partial_\mu$$

where

$$(f \star g)(x) = \frac{1}{(\pi\theta)^2} \int d^2s \, d^2t \; f(x+s) g(x+t) e^{-i2s\Theta^{-1}t}$$

with

$$s\Theta^{-1}t \equiv s^\mu \Theta^{-1}_{\mu\nu} t^\nu \;\; \text{with } \Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \theta \in \mathbb{R}^{+*},$$

and

$$D = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix} \quad \text{with} \quad \partial = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2).$$

The Moyal algebra $\mathcal{A}$ acts on $\mathcal{H}$ as $\pi(f)\psi = \begin{pmatrix} f \star \psi_1 \\ f \star \psi_2 \end{pmatrix}$.

Schrödinger representation

The evaluation at x is not a state of $\bar{\mathcal{A}} = \mathbb{K}$ for $(f^* \star f)(x)$ may not be positive. The pure states of $\bar{\mathcal{A}} = \mathbb{K}$ - i.e. our quantum points - are the vector states in an irreducible representation.

The left regular representation $\mathcal{L}$ is not irreducible, it is a multiple of the Schrödinger representation π_S. Intertwiner:

$$W : h_{mn} \to h_m \otimes h_n \qquad m, n \in \mathbb{N},$$

where the h_{mn}'s are Wigner transition functions (orthonormal basis of $L^2(\mathbb{R}^2)$), h_m's are the eigenfunctions of the quantum h.o. (orthonormal basis of $L^2(\mathbb{R})$).

$$W\mathcal{L}(f)W^* = \pi_S(f) \otimes \mathbb{I} \Longrightarrow \begin{cases} W\mathcal{L}(x_1)W^* = \mathfrak{q} \otimes \mathbb{I} & W\mathcal{L}(z)W^* = \mathfrak{a}^* \otimes \mathbb{I} \\ W\mathcal{L}(x_2)W^* = \mathfrak{p} \otimes \mathbb{I} & W\mathcal{L}(\bar{z})W^* = \mathfrak{a} \otimes \mathbb{I} \end{cases}$$

with $\mathfrak{q}, \mathfrak{p}, \mathfrak{a}, \mathfrak{a}^*$ the position, momentum, creation and annihilation operators.

- The set of pure states of $\overline{\mathcal{A}} = \mathbb{K}$ is thus the set of vector states,

  $$\omega_\psi(f) \doteq \langle \psi, \pi_S(f)\psi \rangle,$$

  where $\psi = \sum_m \psi_m h_m$ is a unit vector in $L^2(\mathbb{R})$.

Spectral distance on the Moyal plane

Eigenstates of the quantum harmonic oscillator,

$$\omega_m(f) \doteq \langle h_m, \pi_S(f) h_m \rangle,$$

Coherent states, i.e. translations of ω_0,

$$\alpha_\kappa \omega_0(f) \doteq \omega_0 \circ \alpha_\kappa(f)$$

where $(\alpha_\kappa f)(x) = f(x+\kappa)$ for any $\kappa \in \mathbb{C} \simeq \mathbb{R}^2$.

Theorem E. Cagnache, F. D'Andrea, P.M., L. Tomassini, J.C. Wallet (2010-11)

1. $d_D(\varphi, \alpha_\kappa \varphi) = |\kappa|$.

2. The spectral distance on the Moyal plane takes all possible value in $[0, \infty]$ (minimal unitization of the Moyal plane is *not* a compact quantum metric space in the sense of Rieffel).

2. The distance between two distinct eigenstates ω_m, ω_n is:

$$d_D(\omega_m, \omega_n) = \sqrt{\frac{\theta}{\sqrt{2}}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

Optimal element

The element in $\mathcal{A}$ that attains the supremum in the spectral distance formula or a sequence $a_n \in \mathcal{A}$ such that $\|[D, a_n]\| \leq 1$ and $\lim\,(\tilde{\varphi}(a_n) - \varphi(a_n)) = d_D(\tilde{\varphi}, \varphi)$.

Between translated states φ and $\alpha_\kappa\varphi$, the optimal element is

$$l_\kappa(z) = \frac{ze^{-i\Xi} + \bar{z}e^{i\Xi}}{\sqrt{2}}, \quad \text{with} \quad \Xi \doteq \operatorname{Arg}\kappa,$$

both on the Euclidean and the Moyal planes. One has

$$[\partial\!\!\!/, l_\kappa] = -i\sqrt{2}\begin{pmatrix} 0 & \bar{\partial} l_\kappa \\ \partial l_\kappa & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & e^{i\Xi} \\ e^{-i\Xi} & 0 \end{pmatrix}$$

so that

$$[\partial\!\!\!/, l_\kappa]^*[\partial\!\!\!/, l_\kappa] = \mathbb{I}.$$

The optimal element $\mathcal{L}(l_0)$ between eigenstates ω_m, ω_n is a solution of

$$[\partial\!\!\!/, \mathcal{L}(l_0)] = -i\begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix},$$

where S is the shift operator. One has, with e_0 is the projection on h_0,

$$\mathbb{I} - [\partial\!\!\!/, \mathcal{L}(l_0)]^*[\partial\!\!\!/, \mathcal{L}(l_0)] = e_0$$

• *Noncommutative geometry of the Moyal plane: translation isometries, Connes distance between coherent states, Pythagoras theorem*,
P.M., L. Tomassini, arXiv:1111.6164.

• *The spectral distance on the Moyal plane*,
E. Cagnache, F. d'Andrea, P. Martinetti, and J.-C. Wallet, J. Geom. Phys. **61** (2011).

• *A view on optimal transport from noncommutative geometry,*
F. D'Andrea, P.M., SIGMA **6** (2010) 057.

• Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry,
R. Wulkenhaar,P.M., J. Math. Phys. **43** (2002) 182-204.

• Length and distance on a quantum space,
P.M., L. Tomassini, Proc. of Sciences **42** (2011).