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RESEARCH PROJECT

Here is a brief list of projects I am currently working on.

- *Wasserstein/ Monge-Kantorovich distance in Noncommutative Geometry*: The Wasserstein distance W in optimal transport theory corresponds to the minimization of a cost function (the Monge problem). Kantorovich showed that the problem is equivalent to maximizing a cost. Rieffel noticed that in case the cost is the geodesic distance, Kantorovich dual formulation of the Wasserstein distance is precisely Connes spectral distance in the commutative case (i.e. for $\mathcal{A} = C_0^\infty(\mathcal{M})$ and as Dirac operator any operator with the same symbol as the Hodge-Dirac operator $d + d^\dagger$, d the exteriore derivative). From this perspective, Connes formula is a generalization to the noncommutative framework of Kantorovich dual formula of the Wasserstein distance.

	commutative case:		**noncommutative case:**
distance supremum:	$d_{d+d^\dagger}$	$\rightarrow$	d_D
	$\uparrow$		$\vert$
Kantorovich duality:	$d_{d+d^\dagger} = W$		$d_D = W_D?$
	$\downarrow$		$\downarrow$
distance infimum:	W with cost d_{geo}		noncommutative cost ?
		$\searrow$	
		métrique de MK metric in free probability	
		$Biane, Voiculescu.$	

A natural question is: what does survive - if anything - of Kantorovich duality in the non commutative framework ? Namely, is there a "noncommutative cost" with an associated Wasserstein distance whose Connes distance d_D would be the dual ? An idea is to view d_D - restricted to the pure state space $\mathcal{P}(\mathcal{A})$ - as a cost function, and define the associated Wasserstein distance W_D on the whole space state of $\mathcal{A}$ (this makes sense as soon as $\mathcal{A}$ is a unital and separable C^*-algebra). It is not difficult to show that $d \leq W_D$. More interesting is to determine when $W_D = d$ on all or part of the space of states.

An alternative comes from free probability. In [1], Biane and Voiculescu discussed an analogue for free probability of the Wasserstein distance (but not of Kantorovich dual formula). It would be extremely interesting to work out a spectral triple such that d_D is the dual of the BV distance.

This would allow to clarify the extend to which Connes definition of the distance is truly a non-commutative generalization of the Wasserstein distance, opening a way to a theory of Optimal Transport in Noncommutative Geometry. It would also be a starting point to re-interpet in the light of the theory of free probability all the results obtained so far on the on the metric aspect of Noncommutative Geometry.

- *The metric aspect of defomation quantization:* here the angle of attack is to generalize the result obtained in the Moyal plane [2, 6] to Rieffel's deformations of algebras. Namely one should consider the action of an arbitrary (Lie) group G (not necessarily $\mathbb{R}^{2N}$) on an arbitrary (Fréchet, or C*) algebra $\mathcal{A}$ (not necessarily the Schwartz functions on $\mathbb{R}^{2N}$).

  Since there is not (yet) a general construction of spectral triples for arbitrary Rieffel deformations, one could start with the existing examples, like the noncommutative torus or the Connes, Dubois-Violette, Landi spheres.

- *Sub-riemannian geometry*: in [4, 5] the obstruction preventing the spectral distance on a $U(n)$-bundle on S^1 to equal the horizontal distance has a topological origin. The holonomy of the connection is non trivial because the base of the bundle is not simply connected. It would be interesting to understand whether the same obstruction occurs when the holonomy comes from the curvature of the connection.

  The Moyal algebra can be viewed as the group C^*-algebra of the Heisenberg group H (modulo a restriction to the unitary representation of H with non-trivial central character, and a Fourier transform). H is also a seminal example of sub-riemannian geometry (the kernel of the horizontal distribution $\Theta = dz - \frac{1}{2}(xdy - ydx)$ in $\mathbb{R}^3$, together with ∂_z, generates the Heisenberg Lie algebra). By computing the spectral distance associated to the covariant Dirac operator defined by Θ, one would obtain a spectral distance on the Heisenberg group. It would be extremely interesting to compare it to the one of the Moyal plane.

- *Pythagoras theorem for the product of spectral triples*: In [3] we show in full generality some Pythagoras inequalities for the product of spectral triples. It would be interesting to understand under which conditions Pythagoras *equality* actually holds. So far equality holds for i) pure states in the product of a manifold by $\mathbb{C}^2$, ii) translated states (pure or not) in the product of the Moyal plane by $\mathbb{C}^2$. So the relevant point seems to be the existence of a geodesic in the space of states, in the sense of a curve $t \in$

$[0,1] \to c(t) \in \mathcal{S}(\mathcal{A})$ such that $d_D(c(t_1), c(t_2)) = |t_1 - t_2| d_D\left(c(t_0), c(t_1)\right)$. In case i), this is the usual Riemannian geodesic on the manifold, in case ii) this is the orbit of the state under the translation action of $\mathbb{R}^2$.

References

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[5] P. Martinetti. Spectral distance on the circle. *J. Func. Anal.*, 255(1575-1612), 2008.

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