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RESEARCH STATEMENTS
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My research is mainly focused around four directions, whose stream line is the understanding of the geometry of space and time:

- the metric aspect of noncommutative geometry;
- applications of noncommutative geometry to physics, in particular the Standard Model of elementary particles, and Noether symetry on noncommutative spacetimes;
- the modular group for von Neumann algebras, its application in algebraic quantum field theory, and its physical interpretation via the thermal time hypothesis of Connes-Rovelli;
- renormalization and Hopf algebras.

I **Noncommutative geometry:** I am particularly interested in the *metric aspect* of Connes theory of spectral triple, and its link with other areas of geometry, like sub-Riemannian geometry or the Wasserstein (or Monge-Kantorovich) distance in the theory of optimal transport.

Recall that given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra with representation π on some Hilbert space $\mathcal{H}$, while D is a densely defined selfadjoint operator on $\mathcal{H}$ generalizing the Dirac (or Atiyah) operator of spin geometry, Connes *spectral distance* on the space of states (i.e. positive linear application $\mathcal{A} \to \mathbb{C}$ with norm 1) $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ is

$$d(\varphi_1, \varphi_2) \doteq \sup_{a \in \text{Lip}_D(\mathcal{A})} |\varphi_1(a) - \varphi_2(a)| \quad \forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathcal{A}), \tag{1}$$

where the D-Lipschitz ball of $\mathcal{A}$ is defined as

$$\text{Lip}_D(\mathcal{A}) \doteq \{a \in \mathcal{A}, \|[D, \pi(a)\| \leq 1\}. \tag{2}$$

My main results on that matter are (in chronological order):

I.1 *Metric aspect of the Standard Model of elementary particles*: In [14, 27] we compute the spectral distance associated to the spectral triple of the Standard Model, and make precise the interpretation of the (now recently discovered) Higgs field as the component of the metric in a discrete internal dimension. The paper also contains various results on the spectral distance for finite dimensional spectral triples.

I.2 *Gauge theory and sub-Riemannian geometry*: In [17] I falsify a "conjecture" regarding the spectral distance encoded by a *covariant Dirac operator*

$$D = -i \sum_{\mu=1}^{\text{dim}M} \gamma^\mu(\partial_\mu + A_\mu)$$

in $SU(n)$-gauge theory made in [5]. Here one chooses $\mathcal{A} = C_0^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$, with $\mathcal{M}$ a Riemannian spin manifold (the γ^μ's are the Dirac matrices, spanning a representation of the Clifford algebra), so that the space of pure states $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ is a trivial $U(n)$-bundle over $\mathcal{M}$, that one equips with a connection (1-form) A_μ. In [17] I show that, contrary to what was expected in [5], the spectral distance on $\mathcal{P}(\mathcal{A})$ is *not* the horizontal distance associated to the connection A_μ (also known as the Carnot-Carathéodory distance). In particular the latter is infinite - by definition - between any two points in $\mathcal{P}(\mathcal{A})$ that cannot be linked by an horizontal path. In terms of foliation, this means that two distinct leaves in the horizontal foliation of $\mathcal{P}(\mathcal{A})$ are at infinite horizontal distance. On the contrary, the spectral distance selects classes of leaves that are at finite distance. The number of this classes is given by the dimension of the holonomy group of the connection A_μ [19]. This opens interesting and yet unexplored links between noncommutative geometry and sub-Riemannian geometry (as well with loop quantum gravity, where the holonomy - viewed as *Wilson loops* - enters the definition of the length operator).

I.3 *Deformation quantization*: We call "Moyal algebra" the noncommutative deformation of the algebra of Schwartz functions on $\mathbb{R}^{2n}$ via a symplectic form Θ (meaning that the usual pointwise product of two functions is replaced by their convolution twisted by Θ). In [3] we compute the distance between a certain class of states of the Moyal algebra (corresponding to the eigenstates of the quantum harmonic oscillator). In [25], we obtain a general result on the Moyal plane (i.e. $n = 1$): the distance between *any* state of the Moyal algebra and *any* of its translation by action of $\mathbb{R}^2$ is precisely the amplitude of the translation. This is an interesting mathematical result by itself (i.e. one of the first times besides the commutative and almost commutative cases that one has an explicit computation of Connes distance on such a large class of states) and from a physical point of view as well, since it gives the distance between the coherent states of the harmonic oscillator.

In the mathematical physics literature there is a common belief that "quantizing the coordinates" yields the emergence of a minimal length. In [23] we show that this issue is more subtle, by elaborating a general framework to compare Connes spectral distance (which has no non-zero lower bound) to the *quantum length* that is defined in various models of quantum spacetime as the spectrum of a suitable length operator, and which does have a non-zero lower bound (like in the Doplicher-Fredenhagen-Robert [DFR] model of 1995 [12]). In [23], using the results of [25], we prove that up to a doubling of the Moyal algebra (that is taking its product by $\mathbb{C}^2$) the DFR quantum length and Connes spectral distance in the Moyal plane are equal on the set of states of optimal localization. Between the eigenstates of the quantum harmonic oscillator, we find that the quantum length and the spectral distance correspond to two different ways of integrating the same noncommutative line element along two different "noncommutative geodesics" (a discrete one for the spectral distance, a continuous one for the quantum length). This gives an interesting interpretation of the formula computed in [3], namely the spectral distance is the Riemann middle-sum approximation of the quantum length.

I.4 *Pythagoras theorem for the product of spectral triples*: In [25], we show that the product of the Moyal space with the spectral triple of $\mathbb{C}^2$ - restricted to coherent states - is orthogonal in the sense of Pythagoras theorem. These results are extended to the product of arbitrary spectral triples in [10]: we prove in full generality some Pythagoras *inequalities* for the product of unital spectral triples, show by example that they are optimal, and work out some non-unital counter-examples inspired by K-homology. As a side result in the commutative case, we also write the corresponding Pythagoras inequalities for the Wasserstein distance.

I.5 *Monge-Kantorovich distance and optimal transport in noncommutative geometry*: In [9] we extend to the locally compact case the equality - first noticed by Rieffel - between Connes distance (in the commutative case) and the Monge-Kantorovich (or Wasserstein) distance in the theory of optimal transport. I also propose in [21] a definition of a Monge-Kantorovich distance W_D in noncommutative geometry, taking the spectral distance on pure states as a cost function.

I.6 *Truncations of noncommutative spaces*: In [8], inspired by regularization in quantum field theory, we study topological and metric properties of spaces in which a cut-off is introduced. A high momentum (short distance) cut-off is implemented by the action of a projection P on the Dirac operator D and/or on the algebra $\mathcal{A}$. This action induces two new distances. We first individuate conditions making them equivalent to the original distance. We then study the Gromov-Hausdorff limit of the set of truncated states, first for quantum metric spaces in the sense of Rieffel, and then for arbitrary spectral triples. In the commutative case, we show that the cut-off induces a minimal length between points, which is infinite if P has finite rank. When P is a spectral projection of D, we work out an approximation of points by non-pure states that are at finite distance from each other. We apply the results to Moyal plane and to the fuzzy sphere, obtained as Berezin quantization of the plane and the sphere. Along the way, we introduce a notion of "state with finite moment of order 1" for noncommutative algebras, give a new proof that the spectral distance between coherent states of Moyal plane is the Euclidean distance between the peaks, and present some new results about Wasserstein distance on the real line and on the circle, including a sharp approximation of the distance between Fejér probability distributions. Finally we discuss discrete approximations of the derivative on the real line: the h-derivative and the q-derivative.

II Physical models of noncommutative spacetimes

In this part of my work, I focused on possible applications to physics of various models of noncommutative geometry (in the sense of Connes), and noncommutative spacetime (in the wider sense of mathematical physics, that is a space whose coordinates do not commute, without worrying about the requirements of spectral triples).

II.1 *Spectral action and the Higgs mass*: In [11] we show how the field σ (linked to right-handed neutrinos), recently introduced "by hand" by Chamseddine and Connes in the spectral action [4] in order to pull back the initial 170 GeV Higgs-mass prediction to the recently observed 126 GeV value, can be obtained in a natural way within the spectral triple framework, Namely, we show how to obtain σ as a fluctuation of the metric, similar to the one yieldings the Higgs field as a "noncommutative companion" to the gauge fields of the Standard Model. This is done starting not from the "Grand Algebra" $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$, instead of the simpler algebra allowed by NCG axioms and the experimental data, namely $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. We show that the reduction of the Grand Algebra to the algebra of the Standard Model is obtained as a symmetry breaking of the spectral action, where the field σ plays a role similar as the one plaid by the Higgs field in the electroweak spontaneous symmetry breaking.

II.2 *Noncommutative gauge theories and matrix models*: In [26] we study the 2-dimensional noncommutative gauge theory defined by the action

$$S_\Omega = \int d^4x \left(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4} \left\{ \mathcal{A}_\mu, \mathcal{A}_\nu \right\}_*^2 + \kappa \mathcal{A}_\mu \star \mathcal{A}_\nu \right),$$

viewed as a functional of the *covariant gauge* $\mathcal{A}_\mu \doteq A_\mu + \frac{1}{2}\tilde{x}_\mu$ where $x_{\mu\nu} \doteq 2\Theta^{-1}_{\mu\nu}x_\nu$ (Θ is the symplectic form defining the Moyal plane, $F_{\mu\nu}$ the field strength). We begin with the $\Omega = 0$ case, where there is no ultra-violet/infra-red mixing. For $\Omega \neq 0$, we expand the action around a symmetric vacuum, and compute the propagator. We obtain a non-local matrix model, with polynomial interactions of degrees 3 and 4. For $\Omega^2 = \frac{1}{3}$, the cinetic term is a Jacobi operator, that we diagonalize. The propagator then comes out as a functional of Chebyschev polynomials of the second kind. 1 point and 2 points correlation functions are computed, and a family of interesting symmetric vacuum is worked out.

II.3 *Noether theory for quantum group deformations of Minkowsky spacetime*: In [2] and [1] we develop a Noether analysis for two kinds of noncommutative spacetimes: θ-Minkowski, characterized by coordinates satisfying the commutation relation $[x_\mu, x_\nu] = \theta_{\mu\nu}$, with $\theta_{\mu\nu}$ a constant matrix, and κ-Minkowski defined by $[x_i, x_0] = ix_i, [x_i, x_j] = 0$. The symmetry of these noncommutative spacetimes are described by quantum group deformations of the Poincaré group (θ-Poincaré and κ-Poincaré). For a non-interacting field theory defined on these noncommutative spacetimes, by imitating the usual Noether theorem, we manage to compute ten Noether currents and charges associated to 10 independent Poincaré-deformed transformations. Interestingly, these ten charges are not in 1-to-1 correspondence with the ten generators of the deformed Poincaré groups. They are linear combinations of the generators, submitted to non trivial constraints (for instance in κ-Minkowski, a boost alone has no associated conserved charge. It must be combined with other generators).

III Modular flow in algebraic quantum field theory

I applied the so called *thermal-time hypothesis* of Connes and Rovelli to various examples in algebraic quantum field theory. Recall that this hypothesis consists in promoting to a real, physical, time flow the abstract mathematical modular flow of Tomita, that appears in the theory of von Neumann algebras.

III.1 *Thermal time hypothesis and the Unruh effect*: In [24] we applied the thermal time hypothesis to the algebra of local observables associated to a double-cone region D in Minkowski spacetime. We obtained some corrections to the Unruh temperature due to the boundedness of the double-cone region (opposed to the unboundedness of the wedge region W, associated to the usual Unruh temperature). Physically, this amounts to pass from an eternal observer (in the wedge W) to an observer with finite life-time (in a double-cone region D). There is a striking effect: as soon as its life-time is finite, an inertial observer experiences an Unruh temperature, whereas for a Wedge observer the Unruh temperature vanishes for a zero acceleration.

In [18], I studied the dependence in the life-time of the observer of our corrected Unruh temperature. In [20] I gave a geometrical interpretation of the correcting factor in terms of the conformal factor associated to the conformal map $W \to D$: the non-vanishing of the temperature for an inertial observer with finite life-time can be traced back to the fact that the conformal factor of the map $W \to D$ is always finite.

III.2 *Geometric modular action in $2D$-boundary conformal field theory*: In [16], we applied the thermal time hypothesis to a Virasoro net of algebras of local observables. Namely we considered the double-cone regions of Minkowski space-time associated to a bi-dimensional conformal field theory with boundary. We found that the modular flow associated to

Longo's ad-hoc state was purely geometrical (as for a CFT in $4D$-Minkowski spacetime), while the modular flow associated to the vacuum combines the previous geometrical action with a term that mixes the components of the field on the edge of the double-cone. This result is particularly interesting for it gives an explicit illustration of Connes theorem, according to which the modular flow defined by two distinct states of the same algebra are unitarily equivalent. Here, we found that the action of Connes cocycle (i.e. the object which implements this unitary equivalence) is purely non-geometrical. This is, in my knowledge, one of the first times that the action of the cocycle has been explicitly computed.

IV **Renormalization and Hopf algebras**

Connes and Kreimer showed in [6, 7] that in perturbative renormalization of quantum field theory, the recursive procedure of extraction of counterterms, in minimal subtraction scheme with dimensional regularization, is identical to a mathematical method of extraction of finite values known as the Birkhoff decomposition. Their result is obtained for the renormalisation of the coupling constant. In [13] we show that the same was true for the renormalisation of the wave function. In [22] we show that the formalism is flexible enough to also encompass the equation of the continuous renormalization group (i.e. the latest also yields a Birkhoff decomposition). These results have been extended in the review paper [15].

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