Habilitation à diriger les recherches

Noncommutative geometry with applications to quantum physics

présentée par

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I Introduction

Understanding the geometry of spacetime is an important question both for physics and mathematics. In Quantum Mechanics space is Euclidean and, following Newton, time is an absolute parameter which is "flowing" everywhere the same. In General Relativity on the contrary, an abstract and absolute time has no meaning and spacetime is suitably described as a pseudo-Riemannian manifold. These two points of view are known to be incompatible and the search for unification has been at the origin of many developments, ever since the birth of Quantum Mechanics. Partial unification is obtained by the Standard Model of elementary particles which combines Quantum Theory and Special Relativity within the framework of Quantum Field Theory, in order to obtain a so far coherent description of three of the four known elementary interactions (electromagnetism, weak and strong interactions). Gravity does not enter the model and one may believe this is not a problem, at least not an emergent question, since each theory is working well in its domain of application and the physical situations in which both quantum and gravitational effects are to be taken into account (e.g. the first instants of the universe after the Big-Bang, or strong gravitational fields and black-holes in astrophysics) are not easily accessible to experiment. Or one may not be satisfied by this dichotomy and rater consider that it reflects a temporary difficulty more than a fundamental obstruction (the history of physics is rich of such examples, where apparent oppositions are solved within a major unifying discovery). Therefore different strategies are explored in order to make gravity compatible with quantum mechanics: one can postulate some structure to fundamental objects, like strings or branes. Or one may separate unification from quantization and try to apply some canonical quantization techniques to the gravitational field, taking into account its originality, namely that in gravity geometry has to be included into the dynamics and does not constitute an a priori fixed framework. This is the program of Loop Quantum Gravity and Spin Foams. Finally there exists an approach in a sense more conservative, which consists in interpreting the data of the big particle accelerators as constraints pointing towards a still unexplored geometrical structure for spacetime. From a mathematical point of view, the idea is to invent - or discover - geometries beyond the scope of Riemannian manifolds, for instance by viewing Riemannian geometry as a particular case - commutative - of a more global theory of geometry that allows a description of "spaces" for which the algebra of functions is non-commutative. A milestone in this direction is Connes obtaining the Standard Model minimally coupled to Euclidean General Relativity from a slightly noncommutative extension of Euclidean spacetime.

The work presented into this habilitation is related to these different thematics, and is organized around the three following subjects.

1. Noncommutative geometry.

I have studied in particular the metric aspect of Connes theory. The main results are the interpretation of the Higgs field as the component of a metric in some "internal" discrete extra-dimension (this result, obtained with Raimar Wulkenhaar, was included in my PhD)[29] and the metric interpretation of gauge fields. In [21,25] it is shown that the latter, seen as connections on a fibre bundle P, not only equip P with the horizontal distance d_H (also called Carnot-Carathéodory distance) well known in sub-Riemannian geometry, but also with a distance whose origin is purely noncommutative. This *spectral distance* d is richer than the horizontal one in the sense that d remains finite between a certain class of leaves of the horizontal foliation of P. This is a rather unexpected result since in [9] it was expected that d and d_H were equal. The obstruction is due to the holonomy of the connection and to have a good control on it, it is useful to study easy examples[25] where P is a $U(n)$-bundle over the circle S^1. Then the connected components of both d and d_H can be worked out, and in low dimension $n = 2$ the spectral distance has been computed explicitly. On a given fiber the result has been extended to $n \geq 2$. In a recent collaboration with Francesco d'Andrea[12] we studied the link, initially noticed by Marc Rieffel, between the spectral distance and the Wasserstein distance of order 1 used in optimal transport theory.

2. Noether analysis in Noncommutative spaces. I am also interested in non-commutative spaces obtained as a deformation of coordinates of Minkowski spacetime. Such spaces appear as effective models in the coupling of matter field with some proposals of quantum gravity in 3 or 4 dimensions[19], but they have their own interest from a mathematical perspective. A first class of deformations is of Lie algebraic kind,

$$[x^\mu, x^\nu] = C^{\mu\nu}_\lambda x_\lambda$$

where $C^{\mu\nu}_\lambda$ is a constant. In collaboration with Amelino-Camelia's group in Rome we have proposed the first steps towards a Noether like analysis in such spaces[2], as well as for twisted deformation of Minkowski spacetime[1],

$$[x^\mu, x^\nu] = \Theta_{\mu\nu}$$

where $\Theta_{\mu\nu}$ is a constant. The main result is a selection principle[3] on the symmetries of those deformed Minkowski spacetimes that can be considered as viable candidates for Noether charges.

3. Physical interpretation of the modular group. The last part of this thesis deals with the physical interpretation of the modular group. The latter is an important tool in the theory of Von Neumann algebras and shows that such algebras are "intrinsically dynamical objects", in that they come equipped with a canonical 1-parameter group of autormorphism which formally looks like the time evolution operator in quantum mechanics. This result, which is at the heart of Connes classification of type 3 factors, is also important in algebraic quantum field theory[14] in order to characterize, via the KMS condition, the states of thermal equilibrium. The idea to use the modular group in quantum gravity to retrieve the notion of proper time at the classical limit (i.e. relativistic but non-quantum) has been stated in a systematic way by Connes and Rovelli[11] under the name of *thermal time hypothesis.* Because of the lack of a well established notion of algebra of observables this hypothesis has not been tested in quantum gravity yet. However together with Carlo Rovelli we checked its validity in quantum field theory. This lead us to an adaptation of the Unruh effect (thermalisation of the vacuum for a uniformly accelerated observer with infinite lifetime[5,4]) to an observer with finite lifetime[28]. We found that the vacuum is a thermal state but whose temperature is not a constant along the trajectory. Moreover because the temperature is proportional to the inverse of the conformal factor of the map that sends wedges of Minkowski spacetime to double-cones[26], this temperatures never vanishes, even for observers with zero acceleration. The physical pertinence of these results have been questioned in [23,24].

II Distances in Noncommutative Geometry

In Connes framework[8] a noncommutative geometry consists in a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra, $\mathcal{H}$ an Hilbert space carrying a representation π of $\mathcal{A}$ and D a selfadjoint operator on $\mathcal{H}$ with compact resolvent such that $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$. In case $\mathcal{A}$ has no unit, one asks instead of the compact resolvent that $\pi(a)(D - \lambda\mathbb{I})^{-1}$ be compact for any $a \in \mathcal{A}, \lambda \notin \text{Sp } a$, with $\mathbb{I}$ the identity in $\mathcal{B}(\mathcal{H})$. Together with a graduation Γ and a real structure J both acting on $\mathcal{H}$, the components of a spectral triple satisfy a set of properties[9,10] providing the necessary and sufficient conditions for 1) an axiomatic definition of Riemannian (spin) geometry in terms of commutative algebra 2) its natural extension to the noncommutative framework.

II.1 The distance formula

Points in noncommutative geometry are recovered as pure states of $\mathcal{A}$, in analogy with the commutative case where the space $\mathcal{P}(C_0(\mathcal{X}))$ of pure states of the algebra of functions vanishing at infinity on some locally compact space $\mathcal{X}$ is homeomorphic to $\mathcal{X}$,

$$x \in \mathcal{X} \iff \omega_x \in \mathcal{P}(C_0(\mathcal{X})) : \omega_x(f) = f(x) \quad \forall f \in C_0(\mathcal{X}). \tag{II.1}$$

Writing $\mathcal{S}(\mathcal{A})$ the set of (non-necessarily pure) states of the (non-necessarily commutative) C^*-closure of $\mathcal{A}$, a distance d between $\rho, \rho' \in \mathcal{S}(\mathcal{A})$ is defined by

$$d(\rho, \rho') \doteq \sup_{a \in \mathcal{A}} \left\{ |\rho(a) - \rho'(a)| \, ; \; \|[D, \pi(a)]\| \leq 1 \right\} \tag{II.2}$$

where the norm is the operator norm on $\mathcal{H}$. Applied to the canonical spectral triple T_E associated to a compact Riemannian manifold $\mathcal{M}$ without boundary, namely

$$\mathcal{A}_E = C^\infty(\mathcal{M}), \; \mathcal{H}_E = L_2(\mathcal{M}, S), \; D_E = -i\gamma^\mu(\partial_\mu + \nabla_\mu) \tag{II.3}$$

with $\mathcal{H}_E$ the space of square integrable spinors, γ^μ the Clifford action of dx^μ on $\mathcal{H}_E$ and ∇_μ the spin connection (when $\mathcal{M}$ is locally compact one considers instead $\mathcal{A} = C_0^\infty(\mathcal{M})$), formula (II.2) between pure states gives back the geodesic distance defined by the Riemannian structure of $\mathcal{M}$,

$$d(\omega_x, \omega_y) = d_{\text{geo}}(x, y). \tag{II.4}$$

For this reason (II.2) appears as a natural extension of the classical distance formula to the noncommutative framework, all the more as it does not involve any notion ill-defined in a quantum framework such as "minimal trajectory between points". In fact d only relies on the spectral properties of the algebra $\mathcal{A}$ and the Dirac operator D, and therefore is called the *spectral distance* in the following.

A connection on a geometry $(\mathcal{A}, \mathcal{H}, D)$ is defined via the identification of $\mathcal{A}$ as a finite projective module over itself (i.e. as the noncommutative equivalent of the section of a vector bundle via Serre-Swan theorem)[9]. It is implemented by substituting D with a *covariant operator*

$$D_A \doteq D + A + JAJ^{-1} \tag{II.5}$$

where A is a selfadjoint element of the set Ω^1 of 1-forms

$$\Omega^1 \doteq \left\{ \pi(a^i)[D, \pi(b_i)] \; ; \; a^i, b_i \in \mathcal{A} \right\}. \tag{II.6}$$

Since $\|[D, \pi(a)]\|$ has no reason to equal $\|[D_A, \pi(a)]\|$, substituting D with D_A induces a change in the metric called a *fluctuation of the metric*. The latest is parametrized by the part of D_A that does not obviously commute with the representation π, namely

$$\mathcal{D} \doteq D + A. \tag{II.7}$$

Spectral triples provide an efficient tool to study spaces whose fine structure goes beyond the scope of classical Riemannian geometry. For instance spacetime as it emerges from the standard model of particle physics is more accurately described as the product of a continuous - called *external* - geometry T_E (II.3) by an internal geometry $T_I = (\mathcal{A}_I, \mathcal{H}_I, D_I)$. The product-spectral triple is

$$\mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_I, \quad \mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_I, \quad D = D_E \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \tag{II.8}$$

where $\mathbb{I}_I$ is the identity operator of $\mathcal{H}_I$ and γ^5 the graduation of $\mathcal{H}_E$. Using (II.6) one computes the corresponding 1-forms and finds they decompose into two pieces,

$$A = -i\gamma^\mu A_\mu + \gamma^5 H \tag{II.9}$$

where A_μ is an $\mathcal{A}_I$-valued skew-adjoint 1-form field over $\mathcal{M}$ and H is a Ω^1_I-valued selfadjoint scalar field. Specifically when the internal algebra is finite dimensional, i.e. $\mathcal{A}_I$ is a direct sum of matrices algebras, then A_μ has value in the Lie algebra of the group of unitaries of $\mathcal{A}_I$ (the gauge group in physics models) and is called the *gauge part* of the fluctuation of the metric. H is called the *scalar part* of the fluctuation.

II.2 Fluctuation of the metric in product of geometries

As a warming up in [16] we computed the distance (II.2) in various examples of finite dimensional geometries. The initial aim was to find explicit formulas for the $\frac{n(n-1)}{2}$ distances between the n pure states of $\mathbb{C}^n$, as functions of the components of the Dirac operator. The hope was to invert these formulas in order to work out a way to associate to any set of distances between n points (i.e. a collection of positive numbers satisfying the triangle inequality pairwise) a corresponding Dirac operator. However it turned out that already for $n = 4$ the distances were roots of polynomial of order 12, letting no chance to write explicit formulas. We also studied spectral triples based on $M_n(\mathbb{C})$ and $M_n(\mathbb{C}) \oplus \mathbb{C}$ for which all distances have been explicitly computed without difficulties.

II.2.1 Distances in product of geometries

Dealing with finite dimensional algebra was a preliminary work in order to address the metric aspect of products of geometries (II.8). First let us note that for any such product, without restricting to $\mathcal{A}_E$ being $C^\infty(\mathcal{M})$, any couple $(\rho_E \in \mathcal{S}(\mathcal{A}_E), \rho_I \in \mathcal{S}(\mathcal{A}_I))$ is a state of $\mathcal{A}$. Moreover the spectral distance associated to the product-spectral triple, once restricted to either $\mathcal{S}(\mathcal{A}_E)$ or $\mathcal{S}(\mathcal{A}_I)$, gives back the distance associated to a single spectral triple.

Proposition II.1 [29] *Let d_E, d_I, d be the spectral distance in T_E, T_I, $T_E \otimes T_I$ respectively. For ρ_E, ρ'_E in $\mathcal{S}(\mathcal{A}_E)$ and ρ_I, ρ'_I in $\mathcal{S}(\mathcal{A}_I)$,*

$$d\left((\rho_E, \rho_I), (\rho_E, \rho'_I)\right) = d_I\left(\rho_I, \rho'_I\right),$$
$$d\left((\rho_E, \rho_I), (\rho'_E, \rho_I)\right) = d_E\left(\rho_E, \rho'_E\right).$$

Computing the cross-distance, i.e. $d((\rho'_E, \rho'_I), (\rho_E, \rho_I))$ for $\rho'_E \neq \rho_E$, $\rho'_I \neq \rho_I$ is more difficult. To make it tractable one restricts to products where at least one of the algebras, say $\mathcal{A}_I$, is a Von Neumann algebra. This allows to consider normal states, that is those states ρ to which is associated a support, namely a projector $s \in \mathcal{A}_I$ such that ρ is faithful on $s\mathcal{A}_I s$. For a pure state ω, being normal implies

$$sas = \omega(a)s \quad \forall a \in \mathcal{A}_I. \tag{II.10}$$

We say that two normal pure states ω_1, ω_2 are in direct sum if $s_1 a s_2 = 0$ for all $a \in \mathcal{A}$. If furthermore the sum $p = \pi_I(s_1) + \pi_I(s_2)$ of their support commutes with D_I, then the cross-distance projects down to a two point-case $\mathcal{A}_E \otimes \mathbb{C}^2$.

Proposition II.2 [29] *For ω_I, ω'_I two normal pure states of $\mathcal{A}_I$ in direct sum whose sum of supports p commutes with D_I,*

$$d\left((\omega_E, \omega_I), (\omega'_E, \omega'_I)\right) = d_e\big((\omega_E, \omega_1), (\omega'_E, \omega_2)\big)$$

where ω_1, ω_2 are the pure states of $\mathbb{C}^2$ and d_e is the distance associated to $T_e \doteq T_E \otimes T_r$ with

$$\mathcal{A}_r \doteq \mathbb{C}^2, \quad \mathcal{H}_r \doteq p\mathcal{H}_I, \quad D_r \doteq pD_Ip\big|_{\mathcal{H}_r}.$$

Note that this proposition remains true for an algebra $\mathcal{A}_I$ on a field $\mathbb{K}$ other than $\mathbb{C}$, assuming that the notion of states is still available. For instance in the standard model one deals with a real algebra $\mathcal{A}_I$.

II.2.2 Scalar fluctuation and the standard model

Let us apply the propositions above to products of geometries that are relevant for physics, for instance the product of "the continuum by the discrete" obtained by taking for T_E the canonical spectral triple (II.3) associated to a Riemannian spin manifold and assuming T_I is built on a finite dimensional algebra $\mathcal{A}_I$, represented on a finite dimensional vector space $\mathcal{H}_I$ with $D_I \in \mathcal{B}(\mathcal{H}_I)$. Let us first consider a scalar fluctuation of the metric, namely formula (II.2) with D substituted by a covariant Dirac operator (II.7) in which the 1-form A only contains a scalar field H (i.e. $A_\mu = 0$). This amounts to take the product of the manifold by an internal geometry

$$T_I^x \doteq (\mathcal{A}_I, \mathcal{H}_I, D_I(x) = D_I + H(x)) \tag{II.11}$$

in which D_I is a non-constant section of $\operatorname{End}\mathcal{H}_I$. $\mathcal{A}_E$ being nuclear, the pure states of $\mathcal{A}_E \otimes \mathcal{A}_I$ are given by couples (ω_x, ω_I) with $\omega_x \in \mathcal{P}(C^\infty(\mathcal{M})), \omega_I \in \mathcal{P}(\mathcal{A}_I)$. In other term $\mathcal{P}(\mathcal{A}) = P \to \mathcal{M}$ is a trivial bundle with fiber $\mathcal{P}(\mathcal{A}_I)$. Proposition II.1 then yields

Proposition II.3 [29] *Let d_{geo} be the geodesic distance in $\mathcal{M}$ and d_x the spectral distance associated to the spectral triple T_I^x. For any pure states ω_x, ω_y of $C^\infty(\mathcal{M})$ and $\omega_I, \omega'_I \in \mathcal{P}(\mathcal{A}_I)$,*

$$\begin{aligned} d((\omega_x, \omega_I), (\omega_x, \omega'_I)) &= d_x(\omega_I, \omega'_I), \\ d((\omega_x, \omega_I), (\omega_y, \omega_I)) &= d_{geo}(x, y). \end{aligned}$$

While proposition II.2 gives

Proposition II.4 [29] *Let ω_1, ω_2 be two normal pure states of $\mathcal{A}_I$ in direct sum such that the sum of their support commutes with $D_H(x)$ for all x. Then*

$$d(x_1, y_2) = L'((0, x), (1, y))$$

where L' is the geodesic distance in the manifold $\mathcal{M}' \doteq [0,1] \times \mathcal{M}$ equipped with the metric

$$\begin{pmatrix} \|R(x)\|^2 & 0 \\ 0 & g^{\mu\nu}(x) \end{pmatrix}$$

in which $g^{\mu\nu}$ is the metric on $\mathcal{M}$ and R is projection on $s_2\mathcal{A}_I$ of the restriction of D_H to $s_1\mathcal{A}_I$.

Proposition II.4 yields an intuitive picture of the spacetime of the standard model. The latest is described by a product $T_E \otimes T_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ suitably represented over the vector space generated by elementary fermions and D_I a finite dimensional matrix that contains the masses of the elementary fermions together with the Cabibbo matrix and neutrinos mixing angles. Through the *spectral action*[6] the scalar fluctuation H further identifies with the Higgs field. From the metric point of view, one finds that all the states of $M_3(\mathbb{C})$ are at infinite distance from one another whereas the states of $\mathbb{C}$ and $\mathbb{H}$ are in direct sum, with support the identity. Hence the model of spacetime that emerges is a two-sheet model, two copies of the manifold, one indexed by the pure state of $\mathbb{C}$, the other one by the pure state of $\mathbb{H}$. The distance between the two sheets coincides with the geodesic distance in a $(\dim \mathcal{M}) + 1$ dimension manifold, and the extra component of the metric is

$$\|R(x)\|^2 = \left(|1 + h_1(x)|^2 + |h_2(x)|^2\right) m_t^2 \tag{II.12}$$

where h_i are the components of the Higgs field and m_t is the mass of the quark top.

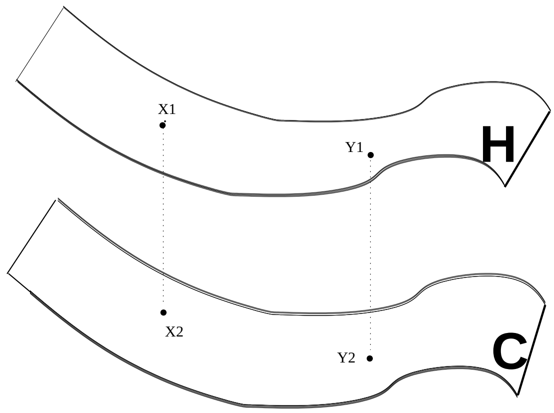


Figure 1: Space-time of the standard model with a pure scalar fluctuation of the metric.

Note that this result has been obtained for a version of the standard model with massless neutrinos. It is expected to be still true in the most recent version[7] that includes massive neutrinos. Work on that matter is in progress.

II.2.3 Gauge fluctuation and the holonomy obstruction

To study a pure gauge fluctuation, i.e. $H = 0, A_\mu \neq 0$, it is convenient to take the product of T_E given in (II.3) by an internal geometry

$$\mathcal{A}_I = M_n(\mathbb{C}), \quad \mathcal{H}_I = M_n(\mathbb{C}), \quad D_I = 0.$$

The set of pure states of $\mathcal{A}$ is a trivial bundle $P \to \mathcal{M}$ with fiber $\mathcal{P}(\mathcal{A}_I) = \mathbb{C}P^{n-1}$. We write

$$\xi_x = (x \in \mathcal{M},\ \xi \in \mathbb{C}P^{n-1}) \tag{II.13}$$

an element of P and s_ξ the support of ξ. The covariant Dirac operator $\mathcal{D}$ of (II.7) is precisely the covariant derivative on this bundle associated to the connection A_μ.

One easily shows that the spectral distance is not greater than the horizontal (or Carnot-Carathéodory[30]) distance d_H defined as the length of the shortest path whose tangent vector is everywhere in the horizontal distribution defined by the connection. In fact d_H plays for fibre bundles the same role as the geodesic distance plays for manifold, so that one expects at first sight that both distance should be equal[9]. However a more careful analysis shows that the holonomy of the connection A_μ is playing a non-trivial role. It is quite obvious that when the holonomy is trivial, that is to say the holonomy group reduces to the identity, then $d = d_H$ on all P. This follows directly from these two lemmas characterizing states at infinite distance from one another,

Lemma II.5 [21] *$d(\xi_x, \zeta_y)$ is infinite if and only if there is a sequence $a_n \in \mathcal{A}$ such that*

$$\lim_{n\to+\infty} \|[D, a_n]\| \to 0, \qquad \lim_{n\to+\infty} |\xi_x(a_n) - \zeta_y(a_n)| = +\infty; \tag{II.14}$$

Lemma II.6 [21] *Let $\xi, \zeta \in \mathbb{C}P^{n-1}$. If there exists a matrix $M \in M_n(\mathbb{C})$ that commutes with the holonomy group at x and such that*

$$Tr(s_\xi M) \neq Tr(s_\zeta M), \tag{II.15}$$

then $d(\omega, \omega') = +\infty$ for any $\omega \in Acc(\xi_x)$, $\omega' \in Acc(\zeta_x)$.

However when the holonomy is non-trivial the set

$$\text{Acc}(\xi_x) \doteq \{p \in P \text{ such that } d_H(\xi_x, p) < +\infty\} \tag{II.16}$$

of states that are accessible from ξ_x by some horizontal curve no longer coincides with the connected component of ξ_x for the spectral distance,

$$\text{Con}(\xi_x) \doteq \{p \in P \text{ such that } d(\xi_x, p) < +\infty\}. \tag{II.17}$$

Since $d \leq d_H$ one has

$$\text{Acc}(\xi_x) \subseteq \text{Con}(\xi_x) \tag{II.18}$$

but the equality has no reason to hold, unless there exists a minimal horizontal curve between ξ_x and p that does not intersect the same orbit of the holonomy group too many times. To be more explicit, let us define

Definition II.7 *Given a curve c in a fiber bundle with horizontal distribution H, we call a c-ordered sequence of K self-intersecting points at p_0 a set of at least two elements $\{c(t_0), c(t_1), ..., c(t_K)\}$ such that for any $i = 1, ..., K$*

$$\pi(c(t_i)) = \pi(c(t_0)), \quad d_H(c(t_0), c(t_i + 1)) > d_H(c(t_0), c(t_i)).$$

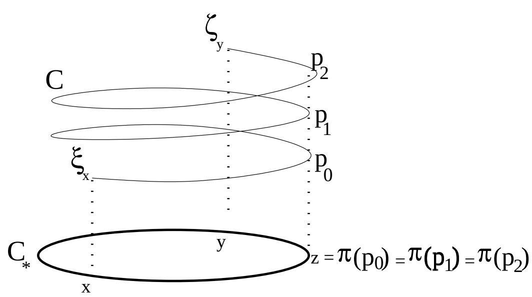


Figure 2: An ordered sequence of self-intersecting points, with $p_i = c(t_i)$.

Then, assuming there exist two pure states for which the spectral distance equals the horizontal one (and is finite), one exhibits a condition applying to each of the points of the minimal horizontal curve between the two states.

Lemma II.8 [21] *Let ξ_x, ζ_y be two points in P such that $d(\xi_x, \zeta_y) = d_H(\xi_x, \zeta_y)$. Then for any $c(t)$ belonging to a minimal horizontal curve c between $c(0) = \xi_x$ and $c(1) = \zeta_y$,*

$$d(\xi_x, c(t)) = d_H(\xi_x, c(t)). \tag{II.19}$$

Moreover, for any such curve there exists an element $a \in \mathcal{A}$ (or a sequence a_n) such that

$$\xi_t(a) = d_H(\xi_x, c(t)) \tag{II.20}$$

for any $t \in [0, 1]$, where ξ_t denotes $c(t)$ viewed as a pure state of $\mathcal{A}$.

Applied to II.7 this lemma immediately gives some necessary conditions for d to equal d_H.

Proposition II.9 *The spectral distance between two points ξ_x, ζ_y in P can equal the Carnot-Carathéodory one only if there exists a minimal horizontal curve c between ξ_x and ζ_y such that there exists an element $a \in \mathcal{A}$, or a sequence of elements a_n, satisfying the commutator norm condition as well as*

$$\xi_{t_i}(a) = d_H(\xi_x, c(t_i)) \quad \textit{or} \quad \lim_{n\to\infty} \xi_{t_i}(a_n) = d_H(\xi_x, c(t_i)) \tag{II.21}$$

for any $\xi_{t_i} = c(t_i)$ in any c-ordered sequence of self-intersecting points.

Given a sequence of K self-intersecting points at p, proposition II.9 puts $K+1$ condition on the n^2 real components of the selfadjoint matrix $a(\pi(p))$. So it is most likely that $d(\xi_x, \zeta_y)$ cannot equal $d_H(\xi_x, \zeta_y)$ unless there exists a minimal horizontal curve between ξ_x and ζ_y such that its projection does not self-intersect more than $n^2 - 1$ times. In fact questioning the equality between d and d_H amounts to the following problem: given a minimal horizontal curve c between two points of a fiber bundle, is there a way to deform c into another horizontal curve c', keeping its length fixed, such that c' has less selfintersecting points than c ? Say differently: can one characterize the minimum number of selfintersecting points in a minimal horizontal curve between two given points? It seems that there is no known answer to this question[30]. It might be possible indeed that in a manifold of dimension greater than 3 one may, by smooth deformation, reduce the number of self-intersecting points of a minimal horizontal curve. But this is certainly not possible in dimension 2 or 1. In order to escape these issues, one can consider a case where there is at most one minimal horizontal curve between two points, namely bundles on the circle S^1.

II.3 Spectral distance on the circle

Consider the trivial $U(n)$-bundle P over the circle of radius 1 with fiber $\mathbb{C}P^{n-1}$. P is the set of pure states of $C^\infty(S^1, M_n(\mathbb{C})) = C^\infty(S^1) \otimes M_n(\mathbb{C})$. The base S^1 being 1-dimensional, the connection 1-form A entering the covariant Dirac operator (II.7) has only one component $A_\mu = -A_\mu^* = A$. Once for all we fix on $\mathcal{H}_I$ a basis of real eigenvectors of iA such that

$$A = i \begin{pmatrix} \theta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \theta_n \end{pmatrix} \tag{II.22}$$

where the θ_j's are real functions on S^1. We let $\mathbb{R}$ mod 2π parametrize the circle and x be the point with coordinate 0.

II.3.1 Connected components

Let us begin by working out the set of accessible states $\text{Acc}(\xi_x)$ defined in (II.16). Any curve c_* in S^1 starting at x is of the form $c_*(t) = t$ with $t \in [0, \tau], \tau \in \mathbb{R}$. Within a trivialization (π, V) and initial condition

$$V(c(0)) = \xi = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \in \mathbb{C}P^{n-1},$$

the horizontal lift $c(\tau) = (c_*(\tau), V(\tau))$ of c_* has components $V_j(\tau) = V_j e^{-i\Theta_j(\tau)}$ where

$$\Theta_j(\tau) \doteq \int_0^\tau \theta_j(t)dt. \tag{II.23}$$

With notations (II.13) the elements of P accessible from $\xi_x = \xi_0$ are thus $\xi_\tau \doteq (c_*(\tau), V(\tau))$, $\tau \in \mathbb{R}$. On a given fiber $\pi^{-1}(c_*(\tau))$ the states accessible from ξ_x are

$$\xi_\tau^k \doteq \xi_{\tau+2k\pi}, \quad k \in \mathbb{Z}. \tag{II.24}$$

Dividing each ξ_τ^k by the irrelevant phase $e^{-ik\Theta_1(2\pi)}$ and writing $\Theta_{ij} \doteq \Theta_i - \Theta_j$, one obtains the orbit H_τ^ξ of ξ_τ under the action of the holonomy group at $c_*(\tau)$, namely

$$H_\tau^\xi \doteq \text{Acc}(\xi_x) \cap \pi^{-1}(c_*(\tau)) = \{ \begin{pmatrix} V_1(\tau) \\ e^{ik\Theta_{1j}(2\pi)} V_j(\tau) \end{pmatrix}, \ k \in \mathbb{Z}; j = 2, ..., n \}.$$

Fiber-wise the set of accessible points H_τ^ξ is thus a subset of the $(n-1)$-torus of $\mathbb{C}P^{n-1}$,

$$T_\xi \doteq \{ \begin{pmatrix} V_1 \\ e^{i\varphi_j} V_j \end{pmatrix}, \ \varphi_j \in \mathbb{R}, \ j = 2, ..., n \}. \tag{II.25}$$

The union on all fibers yields:

Proposition II.10 [25] *$Acc(\xi_x) = \bigcup_{\tau \in [0, 2\pi[} H_\tau^\xi$ is a subset of the n-torus $\mathbb{T}_\xi \doteq S^1 \times T_\xi$.*

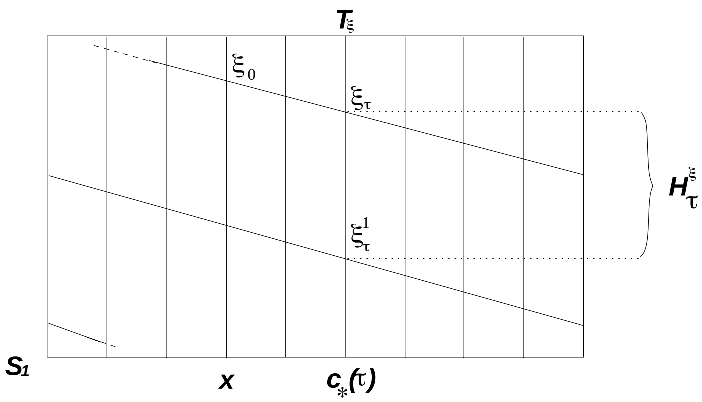


Figure 3: The n-torus $\mathbb{T}_\xi$ with base S^1 and fiber the $(n-1)$-torus T_ξ. The diagonal line is $\text{Acc}(\xi_x = \xi_0) = \{\xi_\tau, \tau \in \mathbb{R}\}$. H_τ^ξ is the orbit of ξ_τ under the holonomy group at $c_*(\tau)$.

Let us now investigate the connected component $\mathrm{Con}(\xi_x)$ of the spectral distance.

Definition II.11 *We say that two directions i, j of T_ξ are far from each other if the components i and j of the holonomy at x are equal, and we write $Far(.)$ the equivalence classes,*

$$Far(i) \doteq \{j \in [1, n] \text{ such that } \Theta_j(2\pi) = \Theta_i(2\pi) \ mod[2\pi]\}. \tag{II.26}$$

Two directions belonging to distinct equivalence classes are said close to each other. We denote n_c the numbers of such equivalence classes and we label them as

$$Far_1 = Far(1),\ Far_p = Far(j_p) \quad p = 2, ..., n_c$$

where $j_p \neq 0$ is the smallest integer that does not belong to $\bigcup\limits_{q=1}^{p-1} Far_q$.

One then gets the connected component of the spectral distance as a subtorus of $\mathbb{T}_\xi$.

Proposition II.12 [25] *$Con(\xi_x)$ is the n_c torus $\mathbb{U}_\xi \doteq \bigcup\limits_{\tau\in[0,2\pi[} U^\xi_\tau$ where $U^\xi_\tau \subset T_\xi$ is the (n_c-1) torus defined by*

$$U^\xi_\tau \doteq \{\begin{pmatrix} V_i(\tau) & \forall i \in Far_1 \\ e^{i\varphi_2} V_i(\tau) & \forall i \in Far_2 \\ \dots & \\ e^{i\varphi_{n_c}} V_i(\tau) & \forall i \in Far_{n_c} \end{pmatrix}, \varphi_j \in \mathbb{R}, j \in [2, n_c]\}. \tag{II.27}$$

Therefore the spectral and the horizontal distances yield two distinct topologies Con and Acc on the bundle of pure states P. Obviously $H^\xi_\tau \subset U^\xi_\tau$ fiber-wise and $\mathrm{Acc}(\xi_x) \subset \mathbb{U}(\xi_x)$ globally, as expected from (II.18). Also obvious is the inclusion of $\mathbb{U}_\xi$ within $\mathbb{T}_\xi$. To summarize the various connected components organize as follow,

$$\mathrm{Acc}(\xi_x) \subset \mathrm{Con}(\xi_x) = \mathbb{U}_\xi \subset \mathbb{T}_\xi \subset P \tag{II.28}$$

or fiber-wise

$$H^\xi_\tau \subset U^\xi_\tau \subset T_\xi \subset \mathbb{C}P^{n-1}. \tag{II.29}$$

The difference between $\mathrm{Acc}(\xi_x)$ and $\mathbb{U}_\xi$ is governed by the irrationality of the connection: when all Θ_{1j} are irrational $\mathrm{Acc}(\xi_x)$ is a dense subset of $\mathbb{U}_\xi$; when all Θ_{1j} are rational $\mathrm{Acc}(\xi_x)$ is a discrete subset. The difference between $\mathbb{U}_\xi$ and $\mathbb{T}_\xi$ is governed by the number of close directions: when all the directions are close to each other (e.g. when the functions θ_i's are constant and distinct from one another) then $U_\xi = T_\xi$, $n_c = n$ and $\mathbb{U}_\xi = \mathbb{T}_\xi$; when all the directions are far from each other, that is to say when the holonomy is trivial, $\mathbb{U}_\xi = S^1 = \mathrm{Acc}(\xi_x)$. To summarize $\mathrm{Con}(\xi_x)$ varies from $\mathrm{Acc}(\xi_x)$ to $\mathbb{T}_\xi$ while $\mathrm{Acc}(\xi_x)$ varies from a discrete to a dense subset of $\mathrm{Con}(\xi_x)$. This is illustrated fiber-wise in low dimension in figure II.3.1.

Note also that $\mathrm{Acc}(\xi_x)$ can be viewed as the unique leaf L^ξ_x in the horizontal foliation of P that contains ξ_x. Then $\mathbb{T}_\xi = \bigcup\limits_{\zeta\in T_\xi} L^\zeta_x$ is the union of all leaves whose intersection ζ with $\pi^{-1}(x)$ has components equal to those of ξ up to phase factors. Meanwhile $\mathbb{U}_\xi = \bigcup\limits_{\zeta\in U_\xi} L^\zeta_x$ where we write $U_\xi = U^\xi_{\tau=0}$ is a subset of the precedent union, with the extra-condition that phase factors corresponding to directions far from each other must be equal. Any two points in different leaves are by definition infinitely Carnot-Carathéodory far from each other. But any two points in leaves belonging to the same $\mathbb{U}_\xi$ are at finite spectral distance. In this sense from the horizontal foliation point of view d is more refined than d_H since it determines some classes of leaves that are at finite distance from each other.

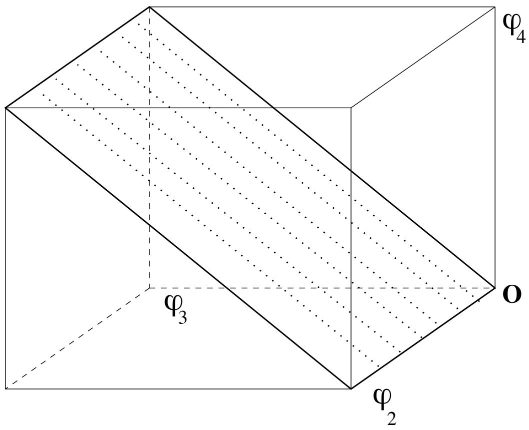


Figure 4: The connected components for the spectral and the horizontal distances on the fiber over x, in case $n = 4$ with directions $3,4$ far from each other. We chose a rational $\Theta_{12}(2\pi)$ and irrational $\Theta_{13}(2\pi) = \Theta_{14}(2\pi)$. The module $|V_i|$ of the components of ξ determine a 3-torus

$$T_\xi = \{\varphi_i \in [0, 2\pi[, i = 2, 3, 4\} \subset \mathbb{C}P^3.$$

The arguments $\operatorname{Arg} V_i$ fixes a point, call it O, in T_ξ. Then:
- $\operatorname{Con}(\xi_x) \cap \pi^{-1}(x)$ is the 2-torus $U_\xi = \{\varphi_2 \in [0, 2\pi[, \varphi_3 = \varphi_4 \in [0, 2\pi[\}$ containing O, represented in the figure by a rectangle with a thick border.
- $\operatorname{Acc}(\xi_x) \cap \pi^{-1}(x)$ is a subset of U_ξ, discrete in φ_2 and dense in $\varphi_3 \sim \varphi_4$, represented in the figure by dot lines inside the rectangle.

II.3.2 A low dimensional example

Let us illustrate propositions II.10 II.12 in the low dimension case $n = 2$. The pure state space of $C^\infty(S^1, M_2(\mathbb{C}))$ is a bundle in sphere over S^1, since the fiber $\mathbb{C}P^1$ identifies to S^2 via

$$\mathbb{C}P^1 \ni \xi = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \mapsto \begin{cases} x_\xi = 2\mathrm{Re}(V_1\overline{V_2}) \\ y_\xi = 2\mathrm{Im}(V_1\overline{V_2}) \\ z_\xi = |V_1|^2 - |V_2|^2 \end{cases} \in S^2. \tag{II.30}$$

Writing $2V_1\overline{V_2} \doteq Re^{i\theta_0}$ one obtains ξ_x as the point in the fiber $\pi^{-1}(x)$ with coordinates

$$x_0 = R\cos\theta_0, \quad y_0 = R\sin\theta_0, \quad z_0 = z_\xi$$

while the accessible points ξ_τ^k defined in (II.24) have coordinates

$$x_\tau^k \doteq R\cos(\theta_0 - \theta_\tau^k), \quad y_\tau^k \doteq R\sin(\theta_0 - \theta_\tau^k), \quad z_\tau^k \doteq z_\xi \tag{II.31}$$

where $\theta_\tau^k \doteq \theta(\tau + 2k\pi)$. In other terms all the ξ_τ^k's are on the circle S_R of radius R located at the "altitude" z_ξ in S^2. Therefore

$$\mathbb{T}_\xi \doteq S^1 \times S_R \tag{II.32}$$

is a 2-dimensional torus (see Figure 5). Assuming the holonomy is not trivial, i.e. that

$$\int_0^{2\pi} A(t)dt$$

is not a multiple of the identity, both directions of T_ξ are closed to each other which means that $\mathbb{U}_\xi = \mathbb{T}_\xi$. Note that when $\Theta(2\pi)$ is irrational $\mathbb{T}_\xi$ is the completion of $\operatorname{Acc}(\xi_x)$ with respect to the Euclidean norm on each S_R.

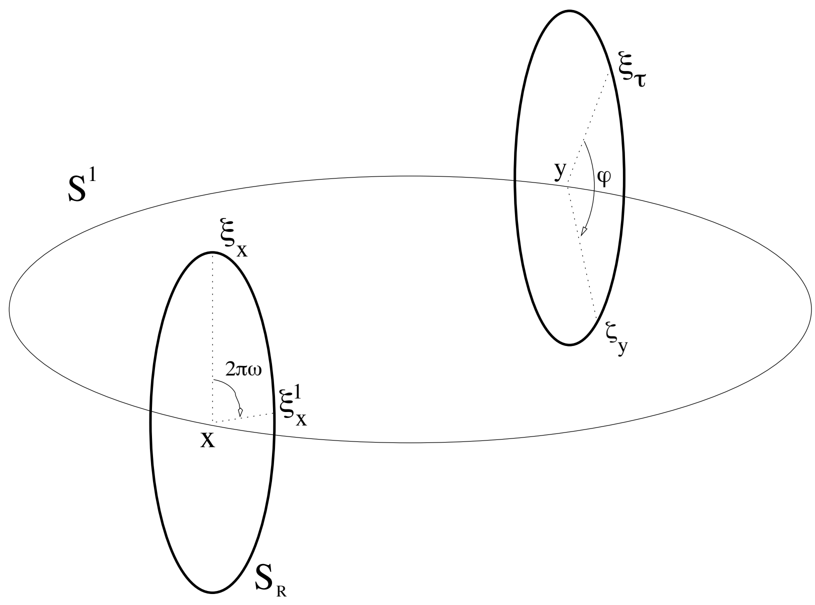


Figure 5: The 2-torus $\mathbb{T}_\xi$, the accessible point ξ_x^1 and an arbitrary pure state ζ_y.

On this low dimensional example one can furthermore explicitly computes all the distance on a given connected component. The first step is to suitably parametrize $\mathbb{T}_\xi$.

Definition II.13 *Given ξ_x in P, any pure state ζ_y in the 2-torus $\mathbb{T}_\xi$ is in one-to-one correspondence with an equivalence class*

$$(k \in \mathbb{N},\ 0 \leq \tau_0 \leq 2\pi,\ 0 \leq \varphi \leq 2\pi) \sim (k + \mathbb{Z},\ \tau_0,\ \varphi - 2\mathbb{Z}\omega\pi) \tag{II.33}$$

with

$$\tau = 2k\pi + \tau_0, \quad \omega \doteq \frac{\Theta_1(2\pi) - \Theta_2(2\pi)}{2\pi} \tag{II.34}$$

such that

$$\zeta_y = \begin{pmatrix} V_1(\tau) \\ e^{i\varphi} V_2(\tau) \end{pmatrix}. \tag{II.35}$$

After a rather lengthy computation, one finds

Proposition II.14 [25] *Let ξ_x be a pure state in P and $\zeta_y = (k, \tau_0, \varphi)$ a pure state in $\mathbb{T}_\xi$. Then either the two directions are far from each other so that $Con(\xi_x) = Acc(\xi_x)$ and*

$$d(\xi_x, \zeta_y) = \begin{cases} min(\tau_0, 2\pi - \tau_0) & when\ \varphi = 0 \\ +\infty & when\ \varphi \neq 0 \end{cases}; \tag{II.36}$$

or the directions are close to each other so that $Con(\xi_x) = \mathbb{T}_\xi$ and

$$d(\xi_x, \zeta_y) = \max_{\mathcal{T}_\pm} H_\xi(T, \Delta) \tag{II.37}$$

where

$$H_\xi(T, \Delta) \doteq T + z_\xi \Delta + RW_{k+1}\sqrt{(\tau_0 - T)^2 - \Delta^2} + RW_k\sqrt{(2\pi - \tau_0 - T)^2 - \Delta^2} \tag{II.38}$$

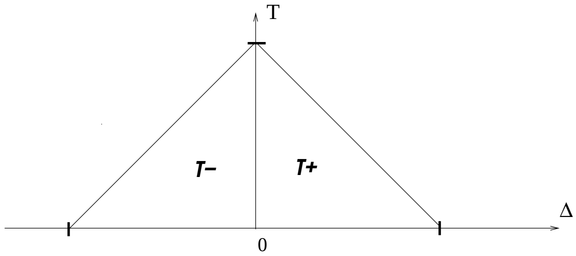
with

$$W_k \doteq \frac{|\sin(k\omega\pi + \frac{\varphi}{2})|}{|\sin \omega\pi|}. \tag{II.39}$$

and the maximum is on one of the triangles

$$\mathcal{T}_{\pm} \doteq T \pm \Delta \leq min(\tau_0, 2\pi - \tau_0) \tag{II.40}$$

with sign the one of z_ξ*.*



Noticing that

Corollary II.15 [25]H_ξ *reaches its maximum either on the segment* $T = 0$ *or on the segment* $T \pm \Delta = \min(\tau_0, 2\pi - \tau_0)$

one obtains that for ξ an equatorial state, i.e. $z_\xi = 0$, the result greatly simplifies

Proposition II.16 [21] $d(\xi_x, \zeta_y) = H_\xi(0,0) = RW_{k+1}\tau_0 + RW_k(2\pi - \tau_0)$.

II.3.3 Distances on the fiber

In the general case $\mathcal{A} = C^\infty(\mathcal{M}, M_n(\mathbb{C}))$ for arbitrary integer $n \in \mathbb{N}$ one can explicitly compute the spectral distance for two pure states on the same fiber. $\mathbb{T}_\xi$ is now a n-torus and instead of (II.33) one deals with equivalence classes of $(n+1)$-tuples

$$(k \in \mathbb{N},\ 0 \leq \tau_0 \leq 2\pi,\ 0 \leq \varphi_i \leq 2\pi) \sim (k + \mathbb{Z},\ \tau_0,\ \varphi_j - 2\mathbb{Z}\omega_j\pi) \tag{II.41}$$

with

$$\omega_j \doteq \frac{\Theta_1(2\pi) - \Theta_j(2\pi)}{2\pi} \qquad \forall j \in [2, n] \tag{II.42}$$

such that ζ_y in $\mathbb{T}_\xi$ writes

$$\zeta_y = \begin{pmatrix} V_1(\tau) \\ e^{i\varphi_j} V_j(\tau) \end{pmatrix} \tag{II.43}$$

where $\tau \doteq 2k\pi + \tau_0$. As soon as $n > 2$ there is no longer correspondence between the fiber of P and a sphere, however in analogy with the equation following (II.30) we write

$$V_j = \sqrt{\frac{R_j}{2}} e^{i\theta_j^0} \tag{II.44}$$

where $R_j \in \mathbb{R}^+$ and $\theta_j^0 \in [0, 2\pi]$. The spectral distance on a given fiber has a simple expression. To fix notation we consider the fiber over x and we identify ξ_x to the triple $(0,0,0)$.

Proposition II.17 [25] *Given a pure state* $\zeta_x = (k, 0, \varphi_j) \in \mathbb{T}_\xi$*, either* ζ_x *does not belongs to the connected component* $\mathbb{U}_\xi$ *and* $d(\xi_x, \zeta_x) = +\infty$ *or* $\zeta_x \in \mathbb{U}_\xi$ *and*

$$d(\xi_x, \zeta_x) = \pi\, Tr(|S_k|) \tag{II.45}$$

where $|S_k| = \sqrt{S_k^* S_k}$ *and* S_k *is the matrix with components*

$$S_{ij}^k \doteq \sqrt{R_i R_j}\, \frac{\sin\left(k\pi(\omega_j - \omega_i) + \frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}. \tag{II.46}$$

In the low dimensional case $n = 2$ this formula again greatly simplifies. Writing $\Xi \doteq 2k\omega\pi + \varphi$ one gets

Proposition II.18 [21] $d(\xi_x, \xi_y) = d(0, \Xi) = 2\pi R W_k = \frac{2\pi R}{|\sin \omega\pi|} \sin \frac{\Xi}{2}$.

It is quite interesting to note that for those points on the fiber which are accessible from ξ_x, namely $\xi_x^k = (k, 0, 0)$ or equivalently $\Xi = \Xi_k \doteq 2k\omega\pi$, the Carnot-Carathéodory distance is $d_H(0, \Xi_k) = 2k\pi$. Hence as soon as ω is irrational one can find close to ξ_x in the Euclidean topology of S_x some ξ_x^k which are arbitrarily Carnot-Carathéodory-far from ξ_x. In other terms d_H destroys the S^1 structure of the fiber. On the contrary the spectral distance keeps it in mind in a rather intriguing way. Let us compare d to the Euclidean distance d_E on the circle of radius $\mathcal{R} \doteq \frac{2R}{|\sin \theta\pi|}$. At the cut-locus $\Xi = \pi$, the two distances are equal but whereas the function $d_E(0, .)$ is not smooth, the noncommutative geometry distance *is* smooth.

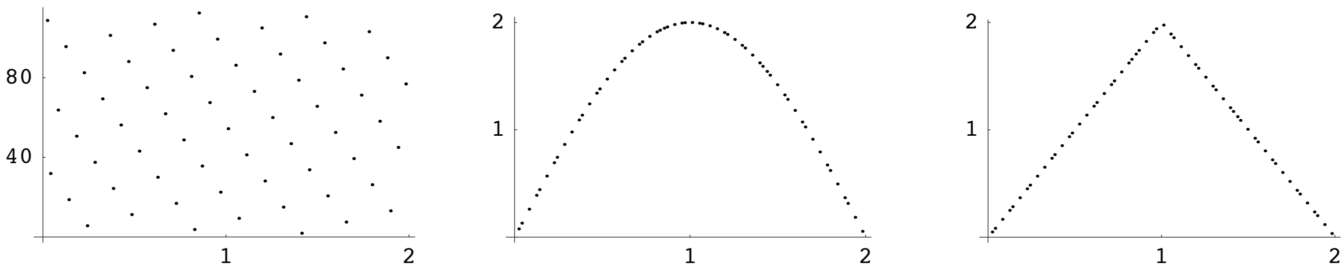


Figure 6: From left to right: $d_H(0, \Xi_k)$, $d(0, \Xi_k)$ and $d_E(0, \Xi_k)$. Vertical unit is $\frac{\pi R}{|\sin \omega\pi|}$, horizontal unit is π.

In [22] we gave the following interpretation of this result: in polar coordinates $C \sin \frac{\phi}{2}$ is the Euclidean length on the cardioid $\frac{C}{4}(1 + \cos \phi)$, so one could be tempted to believe that the fiber equipped with the spectral distance is indeed a cardioid. However one has to be careful with this interpretation. The noncommutative distance d is invariant under translation on the fiber in the sense that the identification of ξ_x with 0 is arbitrary. Identifying 0 to ζ_x with $\zeta \neq \xi$ and $z_\xi = z_\zeta$ would lead to a similar result $d(0, \phi) = C \sin \frac{\phi}{2}$. On the contrary the Euclidean distance on the cardioid is not invariant under translation. To illustrate this difference, let us consider the following cosmological toy model: consider a 1-dimensional universe with topology S^1 and two distinct observers O_1, O_2 located at different points on this S^1 universe. Each of the observers uses a parametrization in which he is the center of the world, namely O_1 uses a parameter Ξ_1, he views himself as occupying the position $\Xi_1 = 0$ and views O_2 occupying a point $\Xi_2 \neq 0$. The same is true for O_2. Assume now the only information the observers can get from the outside world is through the measurement of the distance between themselves and the other points of S^1. Each of the observer measures a function $d_i(\Xi_i) = d(0, \Xi_i)$, $i = 1, 2$. Both agree that the universe they are living in has the shape of a cardioid and both pretend to be localized at this particular point opposite to the "cusp" of the cardioid (see figure 7). Both are equally right and their quarrel is not just a matter of parametrization as on the Euclidean circle. On a cardioid all points are not on the same footing, the cusp and the point opposite to the cusp in particular have unique properties (they are their own image under the axial symmetry σ that leaves the cardioid globally invariant). The very particular position of O_1 according to its own point of view (he is his own image by σ) seems in contradiction with O_2's point of view (for whom O_1 is not σ-invariant). This indicates that the fiber equipped with d is not a Riemannian manifold.

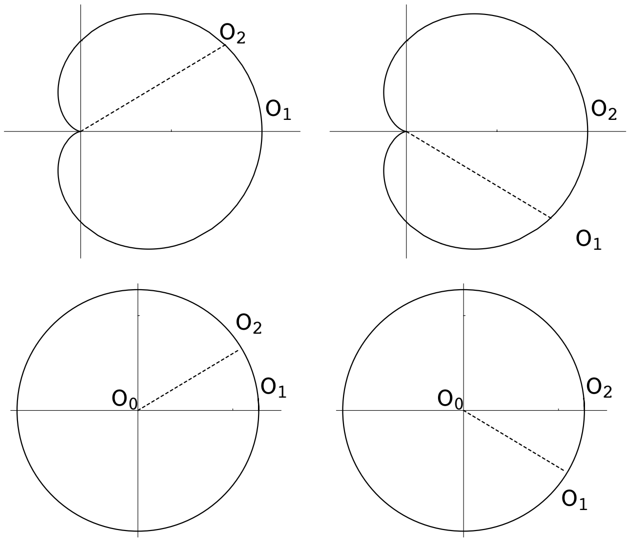
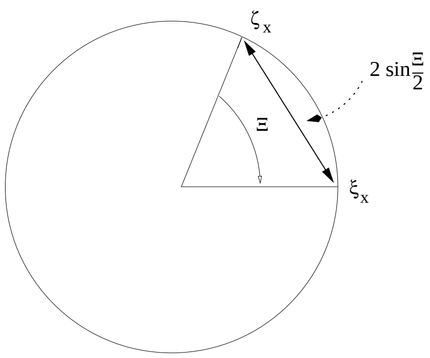


Figure 7: On the left, the world according to O_1, who occupies the position $\Xi_1 = 0$. On the right, the world according to O_2 (located at $\Xi_2 = 0$). At top the world is a cardioid, O_1's and O_2's visions are not compatible with an embedding of the S^1-fiber into a 2-dimensional Riemannian space (the two cardioids do not coincide). At bottom the world is a Euclidean circle. From the intrinsic points of view of the O_i's as well as for an O_0 outside observer embedded into the Euclidean plane, the two objects coincide up to a rotation.

Another interpretation of proposition II.18 is to notice that $\sin\frac{\Xi}{2}$ is the length of the straight segment inside the circle. The spectral distance then appears as the geodesic distance *inside* the disk, in the same way that in the two-sheets model (fig. 1) the spectral distance coincides with a geodesic distance *within* the two sheets.



II.4 Distance between non-pure states & the Monge-Kantorovich problem

The spectral distance (II.2) not only generalizes the geodesic distance to noncommutative algebras, it also extends the distance to objects that are not equivalent to points, namely non-pure states. From the mathematical side some properties of the spectral distance between non-pure states have been questioned by Rieffel in 31 where it is noticed that (II.2) in case $\mathcal{A} = C^\infty(\mathcal{M})$ coincides with a distance well known in optimal transport theory, namely the Wasserstein distance W of order 1,

$$W(\varphi_1, \varphi_2) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left(\int_{\mathcal{X}} f d\mu_1 - \int_{\mathcal{X}} f d\mu_2 \right) \tag{II.47}$$

where φ_i is the state of $C_0^\infty(\mathcal{M})$ given by the probability distribution μ_i,

$$\varphi_i(f) \doteq \int_{\mathcal{M}} f d\mu_i \quad \forall f \in \mathcal{A}. \tag{II.48}$$

The Wasserstein distance is the dual formulation, first obtained by Kantorovich, of Monge ancient "déblais et remblais" problem. Namely given two probability distributions μ_1, μ_2 on some topological space $\mathcal{X}$ and a cost function $c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, the point is to find the minimal cost required to transform the distribution μ_1 into μ_2, that is

$$W(\varphi_1, \varphi_2) \doteq \inf_\pi \int_{\mathcal{X} \times \mathcal{X}} c(x, y) \, d\pi \tag{II.49}$$

where the infimum is over all measures π on $\mathcal{X} \times \mathcal{X}$ with marginals μ_1, μ_2 (i.e. the push-forwards of π through the projections $\mathbb{X}, \mathbb{Y} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $\mathbb{X}(x, y) \doteq x$, $\mathbb{Y}(x, y) \doteq y$, are $\mathbb{X}_*(\pi) = \mu_1$ and $\mathbb{Y}_*(\pi) = \mu_2$). Such measures are called *transportation plans.* Finding the optimal transportation plan, that is the one who minimizes W, is a non-trivial question known as the Monge-Kantorovich problem. In the initial Monge formulation one considers only those transportation plans that are supported on the graph of a *transportation map*, i.e. a map $T : \mathcal{X} \to \mathcal{X}$ such that $T_*\mu_1 = \mu_2$. Namely $W_{\text{Monge}}(\varphi_1, \varphi_2) \doteq \inf_T \int_{\mathcal{X}} c(x, T(x)) \, d\mu_1(x)$.

The interest of Kantorovich[17] generalization is that the infimum in W_{Monge} is not always a minimum: an optimal transportation map may not exist. On the contrary the infimum in (II.49) is a minimum and always coincides with Monge infimum, even when the optimal transportation map does not exist. Moreover when the cost function c is a distance d, (II.49) is in fact a distance on the space of states, called the Kantorovich-Rubinstein distance[18]. To be sure this distance remains finite, it is convenient to restrict to the set $S_1(C_0(\mathcal{X}))$ of states whose moment of order 1 is finite, that is those distributions μ for which the expectation value of the cost function is finite in the sense that for any arbitrary $x_0 \in \mathcal{M}$

$$\mathbb{E}(d(x_0, \circ); \mu) = \int_{\mathcal{X}} d(x_0, x) d\mu(x) < +\infty. \tag{II.50}$$

In [12] we give a proof that the spectral and the Wasserstein distance coincide.

Proposition II.19 [12] *For any $\varphi_1, \varphi_2 \in S_1(C_0^\infty(\mathcal{M}))$ with $\mathcal{M}$ a finite dimensional Riemannian spin manifold*

$$W(\varphi_1, \varphi_2) = d(\varphi_1, \varphi_2) \; .$$

We also find some natural bounds to $d(\varphi_1, \varphi_2)$. An obvious upper bound is the expectation value of the geodesic distance between the two states while, when $\mathcal{M}$ admits a convex Nash embedding into a higher dimensional euclidean manifold, a lower bound is given by the geodesic distance between the mean points $\bar{x}_i$ of the probability distributions

$$d(\bar{x}_1, \bar{x}_2) \leq d(\varphi_1, \varphi_2) \leq \mathbb{E}\big(d\,; \mu_1 \times \mu_2\big) .$$

Restricting to $\mathcal{M} = \mathbb{R}^n$, we explicitly compute the distance between a particular case of states, namely

$$\Psi_{\sigma,x}(f) \doteq \frac{1}{\sigma^n} \int_{\mathbb{R}^n} f(\xi)\psi(\tfrac{\xi-x}{\sigma}) d^n\xi ,$$

for any $x \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^+$, with $\psi \in L^1(\mathbb{R}^n)$ a given density probability, e.g. a Gaussian $\psi(x) = \pi^{-\frac{n}{2}} e^{-|x|^2}$. Noticing that formula above gives back the pure state (the point) δ_x in the $\sigma \to 0^+$ limit, $\Psi_{\sigma,x}$ can be viewed as a "fuzzy" point x_σ, that is to say a wave-packet – characterized by a shape ψ and a width σ – describing the uncertainty in the localization around the point x. The spectral distance between wave packets with the same shape is easily calculated.

Proposition II.20 [12] *The distance between two states $\Psi_{\sigma,x}$ and $\Psi_{\sigma',y}$ is*

$$d(\Psi_{\sigma,x}, \Psi_{\sigma',y}) = \int |x + \sigma\xi - (y + \sigma'\xi)|\, \psi(\xi)\, d^2\xi . \qquad \text{(II.51)}$$

In particular, for $\sigma = \sigma'$ the distance does not depend on the shape ψ:

$$d_D(\Psi_{\sigma,x}, \Psi_{\sigma,y}) = |x - y| .$$

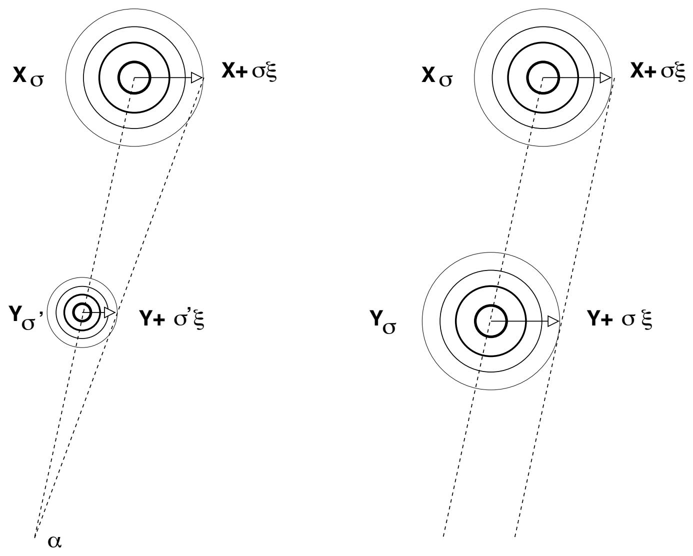


Figure 8: The function h that attains the supremum, in case $\sigma > \sigma'$ and $\sigma = \sigma'$. Dot lines are tangent to the gradient of h.

It is quite interesting to work out the function h that attains the supremum : in case $\sigma \neq \sigma'$, h measures the geodesic distance between $z \in \mathbb{R}^2$ and the point $\alpha \doteq \frac{\sigma' x - \sigma y}{\sigma' - \sigma}$, obtained as the intersection of (x, y) with $(x + \sigma\xi, y + \sigma'\xi)$. In case $\sigma = \sigma'$, α is sent to infinity and h measures the distances between z and an axis perpendicular to (x, y) that does not intersect the segment $[x, y]$. The picture is still valid for pure states, i.e. $\sigma = \sigma' = 0$: h can be taken either as the geodesic distance to any point on (x, y) outside the segment $[x, y]$, or as the distance to any axis perpendicular to (x, y) that does not intersect $[x, y]$.

III Noether analysis on noncommutative spaces

III.1 Deformation of Minkowski space

Besides Connes theory, that provides a noncommutative equivalent to all tools of usual Riemannian differential geometry, physicists are also interested in some less exhaustive approaches to noncommutativity. For instance one may posit a non-commutativity of the coordinates without questioning, at least in a first moment, to what extend geometrical notions like distance or points still make sense. This is a more pragmatic point of view aiming at providing a noncommutative version of some specific tools relevant for the problem under studies (like computing two-point functions within a given field theory). Two of such noncommutative spaces mostly appearing in the literature are deformations of Minkowski spacetime $\mathcal{M}$, either a canonical deformation

$$[x^\mu, x^\nu] = \theta^{\mu\nu} \tag{III.1}$$

with $\theta^{\mu\nu}$ a constant matrix, or a Lie algebraic deformation

$$[x^0, x^j] = \frac{i}{\kappa} x^j, \quad [x^i, x^j] = 0 \tag{III.2}$$

where κ is a constant. The symmetries of these spaces are suitably described by some deformations of the Lie algebra of the Poincaré group G, that we call in the following either θ-Poincaré or κ-Poincaré. Together with Amelino-Camelia group in Rome, we have investigated the Noether currents and charges that might be associated to these deformed symmetries. Our aim was not to provide a full noncomutative version of Noether theorem, but rather to use a Notherian approach in order to give a meaning to notions like “energy” or “momentum” in a noncommutative framework. The hope is to obtain in this way a deformation of the classical relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$ that could be characterized in some experimental context.

Both κ and θ deformations of the Poincaré group are well encoded in a (possibly non-trivial) coproduct Δ on the Lie algebra of generators. The coproduct is defined in such a way that the commutation relation (III.1) or (III.2) are preserved, namely

$$N([x^\mu, x^\nu]) = N(\theta^{\mu\nu}) = 0 \;\text{ or }\; N([x^0, x^j]) = \frac{i}{\kappa} N[x^j],\; N([x^i, x^j]) = 0 \tag{III.3}$$

for any N in the deformed Poincaré algebra. The explicit form of the coproduct depends of course on the noncommutativity under studies, but in any case preserving the commutation relation amounts to deforming the Leibniz rule, since

$$N[fg] = <\Delta N, f \otimes g>$$

equals $N[f]g + fN[g]$ if and only if the coproduct is trivial, $\Delta N = N \otimes \mathbb{I} + \mathbb{I} \otimes N$.

III.2 An attempt for a Noether analysis on noncommutative spacetime

Scalar fields ϕ (or more generally functions) are defined on noncommutative Minkowski spaces through the mapping of ordinary functions on $\mathcal{M}$ via a Weyl map Ω[20],

$$\phi(x) \doteq \int \tilde{\phi}(p)\, \Omega(e^{ipx})\, d^4p \tag{III.4}$$

where $\tilde{\phi}$ is the Fourier transform of a function $\phi \in C^\infty(\mathcal{M})$. There exist different Weyl maps, corresponding to different ordering of the coordinates in the exponential. Each of those yields a different form for the coproduct but the final result turns out to be independent from the ordering.

Let us recall that Noether analysis consists in writing as a divergence the variation $\delta\mathcal{I}$ of an action

$$\mathcal{I} = \int d^4x\,\mathcal{L}(\phi(x)), \tag{III.5}$$

for some Lagrangian density $\mathcal{L}$ on $\mathcal{M}$, under a combined transformation of both the coordinates $x \mapsto x'$ and the field $\phi \mapsto \phi'$ such that the total variation $\phi(x) \to \phi'(x')$ vanishes. Specifically this is obtained by combining the action of $\Lambda = e^{ta} \in G$ on $\mathcal{M}$,

$$x \mapsto \Lambda x \doteq e^{ta}x \quad t \in \mathbb{R}, a \in \mathfrak{g} \tag{III.6}$$

with the scalar action of G on a field,

$$\phi \mapsto \phi' \doteq \phi \circ \Lambda^{-1}. \tag{III.7}$$

Equivalently one can *define* the scalar action of an infinitesimal transformation $\mathbb{I}+\epsilon a$, $\epsilon \ll 1$, as the transformation $\phi \mapsto \phi + \delta\phi$ where

$$\delta\phi = -d\phi \tag{III.8}$$

with

$$d\phi \doteq \epsilon V[f], \tag{III.9}$$

V denoting the vector tangent to the flow generated by a. Thus $\delta\mathcal{I} = 0$ and Noether theorem consists in writing this vanishing variation as a 4-divergence

$$\delta\mathcal{I} = \epsilon^A \int d^4x P_\mu J^\mu_A \phi(x) \tag{III.10}$$

in order to identify Noether currents J^μ_A, whose integration yields conserved charges.

On a deformed noncommutative Minkowski spacetime the strategy is similar. The infinitesimal action of the deformed Poincaré algebra on ϕ makes sense through the Weyl map (III.4), namely

$$N(\phi(x)) = \int \tilde{\phi}(p) N[\Omega(e^{ipx})]\, d^4p. \tag{III.11}$$

As in (III.9) one defines

$$d\phi = \epsilon^A N_A \phi \tag{III.12}$$

where ϵ^A are some infinitesimal coefficients and N_A are generators of the deformed Poincaré algebra. By (III.8) this gives a meaning to the scalar action of $\epsilon^A N_A$ on a field on deformed Minkowski space,

$$\phi \mapsto \phi' = \phi - d\phi. \tag{III.13}$$

The point is then to find an action which remains invariant under this scalar action combined with $x \mapsto \epsilon^A N_A x$, then use the equation of motion together with the commutation rules to put the coefficient ϵ^A on one-side of the integral, so that to obtain an expression similar to (III.10). On can check on some simple examples[27] that a natural requirement in order (III.13) to imply a vanishing of $\delta\mathcal{I}$ is that the transformation satisfies the Leibniz rule,

$$\epsilon^A N_A[fg] = f\epsilon^A N_A[g] + \epsilon^A N_A[f]g. \tag{III.14}$$

Since a single generator does not necessarily satisfies the Leibniz rule, unless it has a trivial coproduct, (III.14) induces some non-trivial relations between the coefficients ϵ^A and the coordinates. As a consequence, some of these coefficients cannot be zero, which indicates that *not all* linear combinations of generators are potential candidates to yield a Noether symmetry. Specifically for θ-Minkowski, writing

$$\epsilon^A N_A = \epsilon^\alpha P_\alpha + \omega^{\mu\nu} N_{\mu\nu} \tag{III.15}$$

with P_α the generators of deformed translations and $N_{\mu\nu}$ those of deformed rotations and boosts, one gets

Proposition III.1 [1] *A linear combination of generators of the θ-deformed Poincaré algebra satisfies the Leibniz rule if*

$$[x^\beta, \omega^{\mu\nu}] = 0 \tag{III.16}$$

$$[x^\beta, \epsilon^\alpha] = -\frac{i}{2}\omega^{\mu\nu}(\theta^\alpha_{[\mu}\delta^\beta_{\nu]} + \theta^\beta_{[\mu}\delta^\alpha_{\nu]}] \tag{III.17}$$

where the brackets denote the anti-symmetrization of the indices, and the latter are lower and raise by the Minkowski signature matrix.

Similarly in κ-Minkowski, writing

$$\epsilon^A N_A = \epsilon^\alpha P_\alpha + \tau^j N_j + \sigma^k R_k \tag{III.18}$$

with R_j the rotation around the j-axis and N_j the boost in the direction j, one obtains

Proposition III.2 [2] *A linear combination of generators of the κ-deformed Poincaré algebra satisfies the Leibniz rule if*

$$\begin{cases} [x^k, \tau^j] = 0 \\ [x^0, \tau^j] = -\frac{i}{\kappa}\tau^j \end{cases} \qquad \begin{cases} [x^k, \sigma^j] = -\frac{i}{\kappa}\epsilon_{jkl}\,\tau^l \\ [x^0, \sigma^j] = 0. \end{cases} \tag{III.19}$$

The physical interpretation of these conditions is quite clear. For θ-Minkowski, (III.17) does not vanish as soon as $\omega^{\mu\nu} \neq 0$, which means that as soon as $\epsilon^A N_A$ contains a pure Lorentz part, then $\epsilon^A N_A$ needs also to contain a translation component. In other terms a pure Lorentz transformation is not a viable candidate for a Noether symmetry. For κ-Minkowski (III.19) indicates that a boost components (i.e. $\tau^j \neq 0$) necessarily implies a rotation part ($\sigma_j \neq 0$). Hence boosts alone are not viable candidates for Noether symmetries.

Under this conditions, one can pursue the Noether analysis. In 1 and 2 we considered the Lagrangian density $\Phi\Box\Phi$, whose variation yields Klein-Gordon equation, and managed to write $\delta\mathcal{I} = 0$ as a 4-divergence. Hence the identification of Noether-like currents[3] together with 10 independent charges. The physical interpretation of these charges, in particular with respect to the dispersion relation, is still under investigation.

As a conclusion, note that the nature of the non-commuting infinitesimal parameter is still not clear. It is tempted to identify ϵ^A to 1-form dx^α, as done in [13]. However one has to be careful not to identify $d\phi$ to the exterior derivative (and so not to assume that $d^2\phi$ is zero whatever ϕ) for $\epsilon^A N_A$ is in fact the *line element* dl of the curve tangent to the action of $a \in \mathfrak{g}$ on $\mathcal{M}$. More precisely by (III.9) the line element writes

$$dl = \|V\| = \frac{\|\epsilon^\mu \partial_\mu\|}{\epsilon} \tag{III.20}$$

where we defined $\epsilon^\mu = \epsilon V^\mu$. Thus it is tempting in the noncommutative case to define[27]

$$dl = \frac{\|\epsilon^A N_A\|}{\epsilon}. \tag{III.21}$$

IV Modular group and time evolution

IV.1 The modular group and Unruh effect in Rindler Wedge

Any Von Neumann algebra $\mathcal{A}$ is, to paraphrase Connes, an intrinsically dynamical object in the sense that it comes equipped with a canonical (up to inner automorphisms) one parameter group of automorphisms G, built from a cyclic and separating vector V in the representation space together with the associated Tomita's operator,

$$S : a \in \mathcal{A} \mapsto Sa \text{ such that } SaV = a^*V \quad \forall a \in \mathcal{A}. \tag{IV.1}$$

It appears that V satisfies with respect to G the same conditions as a thermal equilibrium state with respect to the time evolution, namely the KMS conditions[14]. Hence G is formally a "time evolution" with respect to the state V. This rather abstract assertion can be made concrete within the framework of algebraic quantum field theory[14]. To any field theory on an open region $\mathcal{O}$ of Minkowski spacetime, one associates an algebra of local observables $\mathcal{A}(\mathcal{O})$ which represents the information one can extract from some measurements occurring on $\mathcal{O}$. By the Rhee-Schlieder theorem, as soon as $\mathcal{O}$ has a non-void causal complement, the vacuum vector Ω of a quantum field theory on $\mathcal{M}$ is cyclic and separating for $\mathcal{A}(\mathcal{O})$, hence Ω yields a modular group

$$s \in \mathbb{R} \mapsto \sigma(\mathcal{O})_s \in \text{Aut}(\mathcal{A}(\mathcal{O})) \tag{IV.2}$$

and Ω is KMS with respect to $\sigma(\mathcal{O})$ at temperature -1

$$\omega((\sigma_s A)B) = \omega(B(\sigma_{s-i}A)) \qquad \forall A, B \in \mathcal{A}(\mathcal{O}) \tag{IV.3}$$

where $\omega(.)$ stands for $\langle \Omega, .\Omega \rangle$ and we omit $\mathcal{O}$ to lighten notation. (IV.3) indicates that for an observer whose time-flow is given by $\sigma(\mathcal{O})$ the vacuum Ω has all the properties of an equilibrium state at temperature -1.

To make this statement physically meaningful one has to determine whether $\sigma(\mathcal{O})$ actually corresponds to a concrete physical situation. This is not an easy task since there is a priori no reason that the automorphism $\sigma(\mathcal{O})_s$ can be written as $a \mapsto e^{-iHs}ae^{iHs}$ for some physically meaningful Hamiltonian H. Nevertheless for some open regions $\mathcal{O}$ the identification of $\sigma(\mathcal{O})$ as a physical flow of time is possible: by definition the algebra of local observable $\mathcal{A}(\mathcal{O})$ always carries a representation of the Poincaré group, and it turns out that for some $\mathcal{O}$, e.g wedges or some double-cones, $\sigma(\mathcal{O})$ is precisely generated by a Poincare transformation. Consequently the modular group acquires a geometrical action on spacetime itself and (IV.3) can be interpreted as follows: the vacuum Ω is a thermal state at temperature -1 for an observer whose line of universe coincides with an orbit $H_{\mathcal{O}}$ of the geometrical action of $\sigma_{\mathcal{O}}$ on M. However in order to identify s to the proper time τ of some observer one has to check that ∂_s behaves as a well-defined 4-velocity, namely that it has norm 1. Therefore it is convenient to introduce the normalized flow

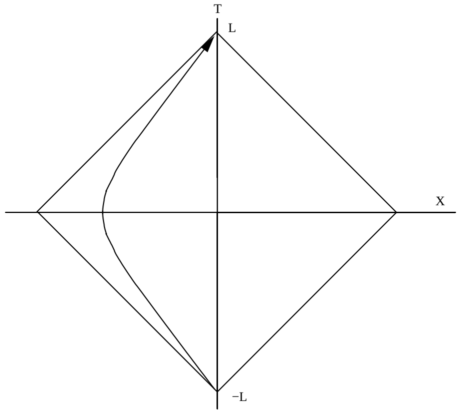
$$\partial_t \doteq -\beta^{-1}\partial_s \text{ where } \beta \doteq \|\partial_s\|\,. \tag{IV.4}$$

One thus obtains that Ω is a thermal state at temperature β^{-1} for an observer with proper time $-\beta s$. The temperature appears as the normalization factor required to make the abstract modular flow ∂_s a physical flow ∂_t. A well known example where this physical interpretation of the modular flow makes sense is the case where $\mathcal{O} = W$ is the Rindler Wedge $|t| < \|\vec{x}\|$. The modular group $\sigma(W)$ is found to be generated by boosts[5,4] whose orbits coincide with the trajectories of observers with a constant acceleration a. To such observers the vacuum appears as thermal state with temperature $T_U = \frac{\hbar a}{2\pi k_B c}$, called the Unruh-Davies temperature.

IV.2 Unruh effect for double-cone

Initially motivated by the *thermal time hypothesis*[11] we investigated the physical interpretation of the modular group for other regions of Minkowski spacetime. Most often there is no reason to expect the modular flow to have a geometric action. Remarkably however this is the case for the modular flow of a diamond region D_L (the causal horizon of a non eternal uniformly accelerated observer, $|t| + \|\vec{x}\| < L$ with L a constant) in the case of a conformally invariant quantum field theory[15]. Identifying the thermal time $-\beta s$ defined in (IV.4) to the

Figure 9: The double-cone and an observer with finite lifetime



proper time τ of the observer, one gets a (inverse) temperature

$$\beta = \|\partial_s\| = |\frac{d\tau}{ds}| \tag{IV.5}$$

Explicit computation yields

Proposition IV.1 [28] *The vacuum of a conformally invariant quantum field theory on Minkowski spacetime is seen by an observer with lifetime $2\tau_0$ and constant acceleration a as a thermal state with temperature*

$$T(\tau) = \frac{\hbar L a^2}{2\pi k_b c^3 \left(\sqrt{1 + \frac{a^2 L^2}{c^4}} - ch\,\frac{a\tau}{c}\right)} \tag{IV.6}$$

where $L = \frac{\sinh a\tau_0}{a}$.

This result generalizes Unruh-Davies temperature T_U to observers with finite lifetime. The main difference with an eternal observer are: i. the temperature is no longer constant along a trajectory but depends on the point on the orbit. ii. it does not vanish for an inertial observer ($a = 0$). From a physical point of view this is coherent with the interpretation of Unruh temperature in term of Hawking temperature for black hole: the edge of the Wedge acts as an horizon for a eternal uniformly accelerated observer and indeed for an observer moving close to the horizon of a eternal black-hole, with acceleration given by the surface gravity, then the Hawking temperature identifies with T_U. For an eternal observer with zero acceleration the edges of the wedge are sent to infinity and the observer has access to the whole of $\mathcal{M}$. Hence it is natural that T_U vanishes. On the contrary for an observer with finite-lifetime, the edge of the double-cone forms an horizon whatever the acceleration, even vanishing. Therefore there is no reason the temperature to vanish. From a mathematical point of view the non-vanishing of the temperature in the inertial case is understood as follows.

Proposition IV.2 [26] *The temperature (IV.6) is proportional to the inverse conformal factor C of the conformal transformation $\varphi : W \to D$, $T(x) = \frac{\hbar a}{2\pi k_b c}\frac{1}{C(x')}$ where $x' \in W$ is the pre-image of $x \in D$ through the map φ.*

Since φ maps an unbounded region W to a bounded region D, the conformal factor which reflects the shrinking of the metric cannot be infinite. Hence it is natural that $T(x)$ never vanishes for $x \in D$.

V Conclusion

As a conclusion, let us mention several works-in progress regarding the various topics presented here.

- Distances in noncommutative geometry: the metric interpretation of the Higgs field in the new version of the standard model (i.e. with massive neutrinos) is in preparation, and it is expected that the same result as in (II.12) holds. More interesting is to incorporate gauge fields in order to perhaps make finite the distance between the states of $M_3(\mathbb{C})$. A more involved mathematical work is to extend the analysis developed in [25] for a bundle with base other than S^1. In particular it would be interesting to study whether a non-trivial holonomy due to curvature rather than topology (i.e. a simply connected base instead of S^1, with a non-flat connection) leads to the same conclusion. Also under studies is the adaptation of the Monge-Kantorovich distance to quantum deformation of Euclidean space, like the Moyal plane.

- Noether analysis: interesting generalization of the results presented here have been recently obtained in Rome for theories with interactions, as well as for finite transformations (as opposed to infinitesimal ones).

- Physical interpretation of the modular group: a paper in collaboration with Henning Rehren and Roberto Long should appear very soon in which we study the temperature for double-cones in 2d conformal quantum field theories with boundary.

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Appendix

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